Paul H. Bezandry · Toka Diagana

Almost Periodic Stochastic Processes

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To our families

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Preface

This book analyzes almost periodic stochastic processes and their applications to various stochastic differential equations, partial differential equations, and difference equations. It is in part a sequel the of authors' recent work [20, 21, 22, 23, 24, 55] on almost periodic stochastic difference and differential equations and has the particularity to be among the few books that are entirely devoted to almost periodic stochastic processes and their applications. The topics treated in it range from existence, uniqueness, boundedness, and stability of solutions to stochastic difference and differential equations.

Periodicity often appears in implicit ways in various natural phenomena. For instance, this is the case when one studies the effects of fluctuating environments on population dynamics. Though one can deliberately periodically fluctuate environmental parameters in controlled laboratory experiments, fluctuations in nature are hardly periodic. Almost periodicity is more likely to accurately describe natural fluctuations [63]. Motivated by this observation, we decided to write this book that is devoted to the study of almost periodic (mild) solutions to stochastic difference and differential equations. Since the beginning of the century, the theory of almost periodicity has been developed in connection with problems related to differential equations, dynamical systems, and other areas of mathematics. The classical books of Bohr [32], Corduneanu [42], Fink [73], and Pankov [151] for instance gave a nice presentation of the concept of almost periodic functions in the deterministic setting as well as pertinent results in the area. Recently, there has been an increasing interest in extending certain classical results to stochastic differential equations in separable Hilbert spaces. This is due to the fact that almost all problems in a real life situation to which mathematical models are applicable are basically stochastic rather than deterministic. Nevertheless, the majority of mathematical methods are based on deterministic models. For instance, the theory of analysis frequently used in deterministic models can often be utilized as a tool to obtain the solutions to stochastic differential equations.

The concept of almost periodicity for stochastic processes was first introduced in the literature by Slutsky [166] at the end of 1930s, who then obtained some reasonable sufficient conditions for sample paths of a stationary process to be almost periodic in the sense of Besicovitch, that is, B^2–almost periodic. A few decades later, two other investigations on the almost periodicity of sample paths followed the pioneer work of Slutsky. Indeed, Udagawa [173] investigated sufficient conditions for sample paths to be almost periodic in the sense of Stepanov, and Kawata [109] studied the uniform almost periodicity of samples paths. Next, Swift [167] extended Kawata's results within the framework of harmonizable stochastic processes. Namely, Swift made extensive use of the concept of uniform almost periodicity similar to the one studied by Kawata to obtain some sufficient conditions for harmonizable stochastic processes to be almost periodic.

This book is divided into eight main chapters. It also offers at the end of each chapter some useful bibliographical notes.

Chapter 1 provides the reader with a detailed and somewhat concise account of basic concepts such as Banach and Hilbert spaces as well as some illustrative examples.

Chapter 2 is devoted to the foundations on operator theory, spectral theory, intermediate spaces, semigroups of operators, and evolution families. Some suitable examples are also discussed. The proofs for several of the classical results are given. The technical Lemma 2.2 (Diagana et al. [52]) and Lemma 2.4 (Diagana [62]) will play a key role throughout the book. Detailed proofs of these technical lemmas are discussed at the very end of Chapter 2.

Chapter 3 develops probabilistic tools needed for the analysis of stochastic problems in the book. It begins with a review of the fundamentals of probability including the notion of conditional expectation, which is very useful in the sequel. This chapter also offers an introduction to the mathematical theory of stochastic processes, including the notion of continuity, measurability, stopping times, martingales, Wiener processes, and Gaussian processes. These concepts enable us to define the so-called Itô integral, the Itô formula, and diffusion processes. An extension of Itô integrals to Hilbert spaces and stochastic convolution integrals are also discussed. An investigation of stochastic differential equations driven by Wiener processes is given at the end of the chapter. Special emphasis will be on the boundedness and stability of solutions.

Chapter 4 introduces and develops the concept of *p-th mean* almost periodicity. In particular, it will be shown that each p-th mean almost periodic process defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ is uniformly continuous and stochastically bounded [132]. Furthermore, the collection of all p-th mean almost periodic processes is a Banach space when it is equipped with its natural norm. Moreover, two composition results for p-th mean almost periodic processes (Theorems 4.4 and 4.5) are established. These two theorems play a crucial role in the study of the existence (and uniqueness) of p-th mean almost periodic solutions to various stochastic differential equations on $L^p(\Omega, \mathscr{H})$ where $\mathscr{H}$ is a real separable Hilbert space.

In Da Prato and Tudor [46], the existence of almost periodic solutions to Eq. (5.3) in the case when the linear operators $A(t)$ are periodic, that is, $A(t+\tau) = A(t)$ for each $t \in \mathbb{R}$ for some $\tau > 0$, was established. In Chapter 5, it goes back to studying the existence of p-th mean almost periodic solutions to the class of nonautonomous stochastic differential equations

$$dX(t) = A(t)X(t)\,dt + F(t, X(t))\,dt + G(t, X(t))\,d\mathbb{W}(t),\ t \in \mathbb{R}, \tag{0.1}$$

where $(A(t))_{t\in\mathbb{R}}$ is a family of densely defined closed linear operators satisfying the well-known Acquistapace–Terreni conditions, $F : \mathbb{R} \times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{H})$ and $G : \mathbb{R} \times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{L}_2^0)$ are jointly continuous satisfying some additional conditions, and $\mathbb{W}(t)$ is a Q-Wiener process with values in $\mathbb{K}$. Application to some N–dimensional parabolic stochastic partial differential equations is also discussed. Moreover, Chapter 5 offers some sufficient conditions for the existence of p-th mean almost periodic solutions to the autonomous counterpart of Eq. (0.1).

Chapter 6 offers sufficient conditions for the existence of p-th mean almost periodic mild solutions for the following classes of stochastic evolution equations with infinite delay

$$\begin{aligned} d\Big[X(\omega,t) + f_1(t, X_t(\omega))\Big] &= \Big[\mathscr{A}X(\omega,t) + f_2(t, X_t(\omega))\Big]dt \\ &\quad + f_3(t, X_t(\omega))d\mathbb{W}(\omega,t),\ \ t \in \mathbb{R},\ \ \omega \in \Omega, \end{aligned} \tag{0.2}$$

where $\mathscr{A} : \mathscr{D} = D(\mathscr{A}) \subset \mathbb{H} \to \mathbb{H}$ is a sectorial linear operator whose corresponding analytic semigroup is hyperbolic, that is, $\sigma(\mathscr{A}) \cap i\mathbb{R}$ is empty, and $f_1 : \mathbb{R} \times \mathbb{H} \to \mathbb{H}_\beta$ $(0 < \alpha < \frac{1}{p} < \beta < 1)$, $f_2 : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$, and $f_3 : \mathbb{R} \times \mathbb{H} \to \mathbb{L}_2^0$ are jointly continuous functions. Chapter 6 also presents some recent results on the existence of p-th mean almost periodic and S^p–almost periodic (mild) solutions to various nonautonomous differential equations using the well-known Schauder fixed point theorem. A few examples are also discussed.

Chapter 7 makes extensive use of abstract results of Chapter 6 to study the existence of square-mean almost periodic solutions to some (non)autonomous second-order stochastic differential equations.

Chapter 8 deals with discrete-time stochastic processes known as random sequences. There, we are particularly interested in the study of almost periodicity of those random sequences and their applications to stochastic difference equations including the so-called Beverton–Holt model.

Almost Periodic Stochastic Processes is aimed at expert readers, young researchers, beginning graduate and advanced undergraduate students, who are interested in the concept of almost periodicity and its applications to stochastic difference and differential equations. The basic background for the understanding of the material presented is timely provided throughout the text.

Last but certainly not least, we are grateful to our families for their continuous support, their encouragement, and especially for their putting up with us during all those long hours we spent away from them, while writing this book.

Paul H. Bezandry and Toka Diagana
October 2010

Contents

Chapter 1
Banach and Hilbert Spaces

In this book, the notations $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ stand respectively for the fields of real, complex, and rational numbers. Further, $\mathbb{N}$ and $\mathbb{Z}$ stand respectively for the set of natural integers and the set of all integers. Throughout the rest of the book, $\mathbb{F}$ will denote a field and unless otherwise stated, when we refer to the field $\mathbb{F}$, we mean $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

This chapter is devoted to the basic material on Banach and Hilbert spaces and their basic properties needed in the sequel. In each section, illustrative examples will be discussed in–depth. Classical examples of Banach and Hilbert spaces are discussed, including quotient spaces, L^p spaces, Hölder spaces C^α, and Sobolev spaces $W^{k,p}$.

1.1 Banach Spaces

1.1.1 Introduction

Banach spaces are one of the most important tools in functional analysis. In this section, we introduce and study those spaces and provide the reader with various examples of Banach spaces. For additional readings on Banach spaces and related issues, we refer the reader to the book by R. E. Megginson [139].

1.1.2 Normed Vector Spaces

Definition 1.1. Let $\mathscr{B}$ be a vector space over the field $\mathbb{F}$. A norm on $\mathscr{B}$ is any mapping $\|\cdot\| : \mathscr{B} \to [0,\infty)$ satisfying: for all $x, y \in \mathscr{B}$ and $\alpha \in \mathbb{F}$,

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha| \,.\, \|x\|$; and

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(iii) $\|x+y\| \leq \|x\| + \|y\|$.

Definition 1.2. A normed vector space is a pair $(\mathscr{B}, \|\cdot\|)$ consisting of a vector space $\mathscr{B}$ and a norm $\|\cdot\|$ defined on it.

Example 1.1. Let $(\mathscr{B}, \|\cdot\|)$ be a normed vector space and let $BC(\mathbb{R}, \mathscr{B})$ denote the vector space of all bounded continuous functions from $\mathbb{R}$ to $\mathscr{B}$.

Define

$$\|\varphi\|_\infty := \sup_{t\in\mathbb{R}} \|\varphi(t)\|.$$

Clearly, $\|\cdot\|_\infty$ is a norm on $BC(\mathbb{R}, \mathscr{B})$, and hance$(BC(\mathbb{R}, \mathscr{B}), \|\cdot\|_\infty)$ is a normed vector space.

Definition 1.3. Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be two norms on $\mathscr{B}$. We say that $\|\cdot\|_0$ is equivalent to $\|\cdot\|_1$ if there exist constants $K_1, K_2 > 0$ such that

$$K_1 \|x\|_0 \leq \|x\|_1 \leq K_2 \|x\|_0, \quad \forall x \in \mathscr{B}.$$

Example 1.2. For the field $\mathbb{F}$, let $\mathbb{F}^d = \mathbb{F} \times \mathbb{F} \times ... \times \mathbb{F}$ (d–copies of $\mathbb{F}$). Now consider the norms on $\mathbb{F}^d$ defined for all $x = (x_1, ..., x_d) \in \mathbb{F}^d$ by

(i) $\|x\|_2 := \left(\sum_{k=1}^{d} |x_k|^2 \right)^{1/2}$,

(ii) $\|x\|_1 := \sum_{k=1}^{d} |x_k|$, and

(iii) $\|x\|_\infty := \max_{1\leq k\leq d} \left|x_k\right|$.

Obviously, $\mathbb{F}^d$ is a normed vector space when it is equipped with any of the previous norms. Moreover, all these norms induce the same topology on $\mathbb{F}^d$.

If $(\mathscr{B}, \|\cdot\|)$ is a normed vector space, then the norm $\|\cdot\|$ induces a metric d on $\mathscr{B} \times \mathscr{B}$ defined by

$$d(x,y) := \|x-y\|, \quad \text{for all} \quad x, y \in \mathscr{B}.$$

The ordered pair $(\mathscr{B}, d)$ then becomes a metric space. Further, the distance d enables us to consider the notion of convergence in $\mathscr{B}$ as follows: a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathscr{B}$ is said to converge to $x \in \mathscr{B}$ as $n \to \infty$ on $\mathscr{B}$ if and only if

$$d(x_n, x) = \|x_n - x\| \to 0 \text{ as } \quad n \to \infty.$$

If the above–mentioned convergence occurs, we then say that the sequence $(x_n)_{n\in\mathbb{N}}$ *converges strongly* to $x \in \mathscr{B}$ and write $x_n \to x$ or $s - \lim_n x_n = x$.

Definition 1.4. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in a normed vector space $(\mathscr{B},\|\cdot\|)$. The sequence $(x_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence if for all $\varepsilon>0$ there exists $N_0\in\mathbb{N}$ such that

$$d(x_n,x_m)=\|x_n-x_m\|<\varepsilon$$

whenever $n,m\geq N_0$.

Proposition 1.1. *If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a normed vector space $(\mathscr{B},\|\cdot\|)$, then the following hold:*

(i) If $(x_n)_{n\in\mathbb{N}}$ contains a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges to some $x\in\mathscr{B}$, then the whole sequence $(x_n)_{n\in\mathbb{N}}$ converges to x.

(ii) There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$\left\|x_{n_k}-x_{n_{k+1}}\right\|\leq\frac{1}{2^k}$$

for $k=1,2,\ldots$.

Proof. (i) Let $\varepsilon>0$ and choose N_0 such that $k\leq N_0$ with $\left\|x_{n_k}-x\right\|<\frac{\varepsilon}{2}$ and M_0 such that $n,m\leq M_0$ with $\|x_n-x_m\|<\frac{\varepsilon}{2}$. Let $N_1>N_0$ with $N_1>M_0$. Now for $n>N_1$, then

$$\|x_n-x\|\leq\|x_n-x_{N_1}\|+\|x_{N_1}-x\|<\varepsilon.$$

(ii) Let N_1 such that for all $n,m\geq N_1$, we have $\|x_n-x_m\|<\frac{1}{2}$. Similarly, let $N_2>N_1$ such that for all $n,m\geq N_2$,

$$\|x_n-x_m\|<\frac{1}{2^2}.$$

Proceeding as previously, one obtains the desired result.

Definition 1.5. A metric space $(\mathscr{B},d)$ is said to be complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges to some $x\in\mathscr{B}$.

Remark 1.1. Note that if $\mathscr{B}$ is a normed vector space, then there exists a complete normed vector space $\tilde{\mathscr{B}}$ such that $\mathscr{B}$ is a dense subset of $\tilde{\mathscr{B}}$. Up to isometry, the normed vector space $\tilde{\mathscr{B}}$ is unique.

Definition 1.6. A normed vector space $(\mathscr{B},\|\cdot\|)$ is said to be a *Banach space* if it is complete.

In what follows, we provide the reader with a few examples of Banach spaces encountered in the literature including the so-called quotient Banach space.

1.1.3 Examples of Banach Spaces

Example 1.3. (Quotient Banach Space) Let $(\mathscr{B}, \|\cdot\|)$ be a normed vector space and $\tilde{\mathscr{B}} \subset \mathscr{B}$ be a closed subspace. The cosets of $\mathscr{B}$ consist of all sets $[x] = x + \tilde{\mathscr{B}},\ x \in \mathscr{B}$.

Set $\mathscr{B}/\tilde{\mathscr{B}} = \{[x] : x \in \mathscr{B}\}$. Observe that $[x] = [y]$ if and only if $x - y \in \tilde{\mathscr{B}}$. Clearly, one can endow the set $\mathscr{B}/\tilde{\mathscr{B}}$ with a vector space structure as follows:

$$[x+y] = [x] + [y], \quad [\lambda x] = \lambda [x].$$

The space $\mathscr{B}/\tilde{\mathscr{B}}$ is called quotient space.

Lemma 1.1. *The mapping* $\|\cdot\| : \mathscr{B}/\tilde{\mathscr{B}} \to [0,\infty)$ *defined by*

$$\|[x]\| = \left\|x + \tilde{\mathscr{B}}\right\| = d(x, \tilde{\mathscr{B}}) = \inf_{y \in \tilde{\mathscr{B}}} \|x - y\|$$

for each $[x] \in \mathscr{B}/\tilde{\mathscr{B}}$, *is a norm on* $\mathscr{B}/\tilde{\mathscr{B}}$.

Proof. Let $x, y \in \mathscr{B}$ and $\alpha \in \mathbb{F}$. Using the closedness of $\tilde{\mathscr{B}}$ it follows that $d(x, \tilde{\mathscr{B}}) = 0$ if and only if $x \in \tilde{\mathscr{B}}$. Consequently, $\|[x]\| = \left\|x + \tilde{\mathscr{B}}\right\| = 0$ if and only if $[x] = x + \tilde{\mathscr{B}} = 0 + \tilde{\mathscr{B}}$.

If $\alpha \neq 0$, then

$$\|\alpha[x]\| = \left\|\alpha(x + \tilde{\mathscr{B}})\right\| = d(\alpha x, \tilde{\mathscr{B}}) = d(\alpha x, \alpha\tilde{\mathscr{B}}) = |\alpha|\, d(x, \tilde{\mathscr{B}}) = |\alpha|\, \|[x]\|.$$

Of course, if $\alpha = 0$, then $\left\|0(x + \tilde{\mathscr{B}})\right\| = \left\|0 + \tilde{\mathscr{B}}\right\| = 0 = |0| \left\|x + \tilde{\mathscr{B}}\right\|$.

It remains to prove the triangle inequality. For that, let $x_1, y_1 \in \tilde{\mathscr{B}}$, then

$$\begin{aligned}
\left\|(x + \tilde{\mathscr{B}}) + (y + \tilde{\mathscr{B}})\right\| &= \left\|(x+y) + \tilde{\mathscr{B}}\right\| \\
&\leq \|x + y + x_1 + y_1\| \\
&\leq \|x + x_1\| + \|y + y_1\|,
\end{aligned}$$

and hence

$$\left\|(x + \tilde{\mathscr{B}}) + (y + \tilde{\mathscr{B}})\right\| \leq \left\|x + \tilde{\mathscr{B}}\right\| + \left\|y + \tilde{\mathscr{B}}\right\|.$$

Theorem 1.1. *If* $\mathscr{B}$ *is a Banach space and if* $\tilde{\mathscr{B}} \subset \mathscr{B}$ *is a closed subspace, then the quotient space* $(\mathscr{B}/\tilde{\mathscr{B}}, \|\cdot\|)$ *defined above is a Banach space.*

Proof. Let $(Y_n)_{n\in\mathbb{N}} \subset \mathscr{B}/\tilde{\mathscr{B}}$ be a Cauchy sequence. Using Proposition 1.1(ii), it follows that it is enough to assume that $\|Y_{n+1} - Y_n\| < \frac{1}{2^n}$ for $n = 1, 2, \ldots$. Let $y_1 \in Y_1$ and choose $z_2 \in Y_2 - Y_1$ such that $\|z_2\| \leq 2\|Y_2 - Y_1\|$. Setting $y_2 = z_2 + y_1$ it follows that $\|y_2 - y_1\| \leq 2\|Y_2 - Y_1\|$ and $y_1 \in Y_1$ and $y_2 \in Y_2$. Proceeding as previously it follows there exists a sequence $y_n \in Y_n$ such that

$$\|y_{n+1} - y_n\| \leq 2\|Y_{n+1} - Y_n\|$$

for $n = 1, 2, \ldots$.

Now for $m > n$, we obtain that

$$\begin{aligned} \|y_m - y_n\| &= \left\| \sum_{k=n}^{m} (y_{k+1} - y_k) \right\| \\ &\leq \sum_{k=n}^{m} \|y_{k+1} - y_k\| \\ &\leq 2 \sum_{k=n}^{m} \|Y_{k+1} - Y_k\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Therefore $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{B}$ and hence converges to some $y \in \mathscr{B}$ as $\mathscr{B}$ is a Banach space. Setting $Y = [y]$ it follows that

$$\|Y_n - Y\| = \|[y_n] - [y]\| = \|[y_n - y]\| \leq \|y_n - y\| \to 0$$

as $n \to \infty$.

Example 1.4. (l^p-Spaces) Let $(\mathscr{B}, \|\cdot\|)$ be a Banach space over the field $\mathbb{F}$. Define, for $1 \leq p \leq \infty$, the space $l^p(\mathscr{B})$ to be the set of all $\mathscr{B}$–valued sequences $x = (x_n)_{n\in\mathbb{N}}$ such that $\|x\|_p < \infty$, where

$$\|x\|_p := \left(\sum_{m\in\mathbb{N}} \|x_m\|^p \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty \tag{1.1}$$

and

$$\|x\|_\infty := \sup_{m\in\mathbb{N}} \|x_m\| \quad \text{if } p = \infty. \tag{1.2}$$

Theorem 1.2. *Let $1 \leq p \leq \infty$. The normed vector space $(l^p(\mathscr{B}), \|\cdot\|_p)$ defined above is a Banach space.*

Proof. We only provide the proof for the case $1 \leq p < \infty$. The proof for the case $p = \infty$ will be left as an exercise. So suppose $1 \leq p < \infty$. Let $x^n = (x^n_m)_{m\in\mathbb{N}} \in l^p(\mathscr{B})$ be a Cauchy sequence for $\|\cdot\|_p$. Therefore, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|x^n - x^m\|_p^p = \sum_{k\in\mathbb{N}} \|x^n_k - x^m_k\|^p < \varepsilon^p \tag{1.3}$$

whenever $n, m \geq n_0$.

Now since

$$\|x^n_k - x^m_k\|^p \leq \|x^n - x^m\|_p^p < \varepsilon^p$$

for all $k \in \mathbb{N}$, it follows that $\|x^n_k - x^m_k\| < \varepsilon$ whenever $n, m \geq n_0$, for each $k \in \mathbb{N}$, and hence $(x^n_k)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathscr{B}$. Since $\mathscr{B}$ is complete, there exists $x_k \in \mathscr{B}$ such that $\|x_k^n - x_k\|$ is small enough whenever n is large enough. From the previous observations, one defines the sequence $x = (x_k)_{k\in\mathbb{N}}$ where

$$x_k := \lim_n x_k^n.$$

Now using the fact that Cauchy sequences are bounded, for each $N \in \mathbb{N}$,

$$\sum_{k=0}^{N} \|x_k\|^p = \lim_{n\to\infty} \sum_{k=0}^{N} \|x_k^n\|^p \leq \sup_{n\in\mathbb{N}} \|x^n\|_p^p = M < \infty.$$

Hence $x \in l^p(\mathscr{B})$.

From (1.3) it follows that for all $N \in \mathbb{N}$,

$$\sum_{k=0}^{N} \|x_k^n - x_k^m\|^p < \varepsilon^p$$

for large n,m. Letting m and N go to ∞ respectively in the previous inequality it follows

$$\sum_{k=0}^{\infty} \|x_k^n - x_k\|^p < \varepsilon^p$$

and hence $\|x^n - x\|_p \leq \varepsilon$, which completes the proof.

Example 1.5. ($L^p(\mathscr{O})$–Spaces) Let $\mathscr{O} \subset \mathbb{R}^n$ be an arbitrary domain and let p be a positive real number. Define the space $L^p(\mathscr{O})$ to be the class of all (Lebesgue) measurable functions $u : \mathscr{O} \mapsto \mathbb{C}$ such that

$$\|u\|_p := \left(\int_{\mathscr{O}} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty. \tag{1.4}$$

This is understood to mean that in $L^p(\mathscr{O})$, one identifies functions, which are equal almost everywhere (a.e.) on $\mathscr{O}$.

Similarly, one defines $L^\infty(\mathscr{O})$ to be the space of all (Lebesgue) measurable functions $u : \mathscr{O} \to \mathbb{C}$ such that

$$\|u\|_\infty := \text{ess sup}\,\{|u(x)| : x \in \mathscr{O}\} < \infty. \tag{1.5}$$

Clearly, for each $1 \leq p \leq \infty$, $(L^p(\mathscr{O}), \|\cdot\|_p)$ is a normed vector space. (Warning: $\|\cdot\|_p$ is not a norm when $0 < p < 1$.)

Let us recall some basic properties of $(L^p(\mathscr{O}), \|\cdot\|_p)$.

Proposition 1.2. *(Hölder's Inequality) Let $1 \leq p,q \leq \infty$ with $p^{-1} + q^{-1} = 1$. If $u \in L^p(\mathscr{O})$ and $v \in L^p(\mathscr{O})$, then $u.v \in L^1(\mathscr{O})$. Moreover,*

$$\int_{\mathscr{O}} |uv(x)| \, dx \leq \|u\|_p \,.\, \|v\|_q \,.$$

The proof of Proposition 1.2 makes use of the following classical result.

Lemma 1.2. *Let a,b be nonnegative real numbers. If $1 < p,q < \infty$ with $p^{-1}+q^{-1} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. (Proposition 1.2) First of all, note that the cases $p = \infty$ or $q = \infty$ are trivial. Consequently, we suppose that $1 < p,q < \infty$. It is enough to suppose that $\|u\|_p \neq 0$ and $\|v\|_q \neq 0$. Indeed, if $\|u\|_p = 0$ for instance, we obtain $u(x)v(x) = 0$ a.e. which yields that Hölder's Inequality holds.

Letting $a = \dfrac{|u(x)|}{\|u\|_p}$ and $b = \dfrac{|v(x)|}{\|v\|_q}$ and using Lemma 1.2 it follows that

$$\frac{|u(x)|}{\|u\|_p} \cdot \frac{|v(x)|}{\|v\|_p} \leq \frac{|u|^p(x)}{p\|u\|_p^p} + \frac{|v|^q(x)}{q\|u\|_q^q}$$

which, by integration on $\mathscr{O}$, yields

$$\int_{\mathscr{O}} \frac{|u(x)|}{\|u\|_p} \cdot \frac{|v(x)|}{\|v\|_p} dx \leq \int_{\mathscr{O}} \frac{|u|^p(x)}{p\|u\|_p^p} dx + \int_{\mathscr{O}} \frac{|v|^q(x)}{q\|u\|_q^q} dx = p^{-1} + q^{-1} = 1.$$

Therefore

$$\int_{\mathscr{O}} |uv| dx \leq \|u\|_p \,.\, \|v\|_p.$$

Proposition 1.3. *(Minkowski's Inequality) Let $1 \leq p < \infty$. If $u,v \in L^p(\mathscr{O})$, then $u+v \in L^p(\mathscr{O})$. Moreover,*

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p .$$

Proof. Let q be such that $p^{-1}+q^{-1} = 1$. Using Hölder's Inequality it follows that

$$\begin{aligned}
\int_{\mathscr{O}} |u(x)+v(x)|^p dx &\leq \int_{\mathscr{O}} |u(x)+v(x)|^{p-1} \,.\, |u(x)| dx + \int_{\mathscr{O}} |u(x)+v(x)|^{p-1} \,.\, |v(x)| dx \\
&\leq \Big(\int_{\mathscr{O}} |u(x)+v(x)|^{q(p-1)} dx\Big)^{1/q} \Big(\int_{\mathscr{O}} |u(x)|^p\Big)^{1/p} \\
&+ \Big(\int_{\mathscr{O}} |u(x)+v(x)|^{q(p-1)} dx\Big)^{1/q} \Big(\int_{\mathscr{O}} |v(x)|^p\Big)^{1/p} \\
&= \Big(\int_{\mathscr{O}} |u(x)+v(x)|^p dx\Big)^{1/q} \Big(\int_{\mathscr{O}} |u(x)|^p\Big)^{1/p} \\
&+ \Big(\int_{\mathscr{O}} |u(x)+v(x)|^p dx\Big)^{1/q} \Big(\int_{\mathscr{O}} |v(x)|^p\Big)^{1/p} \\
&= \Big(\int_{\mathscr{O}} |u(x)+v(x)|^p dx\Big)^{1/q} \Big(\|u\|_p + \|v\|_p\Big)
\end{aligned}$$

which yields

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p .$$

Proposition 1.4. *Suppose* $mes(\mathscr{O}) := \int_{\mathscr{O}} 1dx < \infty$ *and that* $1 \leq p \leq q \leq \infty$. *If* $u \in L^p(\mathscr{O})$, *then* $u \in L^q(\mathscr{O})$. *Moreover,*

$$\|u\|_p \leq (mes(\mathscr{O}))^{\frac{1}{p}-\frac{1}{q}} \,.\, \|u\|_q.$$

Proof. The proof is left as an exercise.

The space $(L^p(\mathscr{O}), \|\cdot\|_p)$ is a Banach space for each $1 \leq p \leq \infty$. If $1 \leq p < \infty$ with $p^{-1} + q^{-1} = 1$, then the (topological) dual of $(L^p(\mathscr{O}), \|\cdot\|_p)$ is $(L^q(\mathscr{O}), \|\cdot\|_q)$. Indeed, for each function $g \in L^q(\mathscr{O})$, we define a continuous linear functional Φ_g on $L^p(\mathscr{O})$ by setting

$$\langle \Phi_g, f \rangle := \int_{\mathscr{O}} f(x)\overline{g(x)}\, dx, \quad \forall f \in L^p(\mathscr{O}).$$

Conversely, it can be shown that every bounded linear functional on $L^p(\mathscr{O})$ is of the form Φ_h, where $h \in L^q(\mathscr{O})$ with $p^{-1} + q^{-1} = 1$.

For $1 \leq p \leq \infty$, we also define $L^p_{loc}(\mathscr{O})$ to be the collection of all measurable functions u defined on $\mathscr{O}$ such that $u \in L^p(\mathscr{O}')$ for any compact subset $\mathscr{O}' \subset \mathscr{O}$.

One says that $(u_n)_{n\in\mathbb{N}} \in L^p_{loc}(\mathscr{O})$ converges to u as $n \to \infty$ in $L^p_{loc}(\mathscr{O})$ provided that $\|u_n - u\|'_p \to 0$ as $n \to \infty$ for any $\mathscr{O}' \subset \mathscr{O}$ compact subset, where $\|\cdot\|'_p$ is the norm of $L^p(\mathscr{O}')$. (Warning: Although $L^p_{loc}(\mathscr{O})$ is a topological vector space, it is not a Banach space.)

Remark 1.2. Let $(\mathscr{B}, \|\cdot\|)$ be a Banach space. As above, one defines $L^p(\mathbb{R}, \mathscr{B})$ for $1 \leq p \leq \infty$ as the class of all (Lebesgue) measurable functions $u : \mathbb{R} \mapsto \mathscr{B}$ such that

$$\|u\|_\infty := \text{ess sup}\{\|u(x)\| : x \in \mathbb{R}\} < \infty \quad (p = \infty) \tag{1.6}$$

and

$$\|u\|_p := \left(\int_{\mathbb{R}} \|f(x)\|^p\, dx\right)^{\frac{1}{p}} < \infty \quad (1 \leq p < \infty). \tag{1.7}$$

The space $L^p(\mathbb{R}, \mathscr{B})$ equipped with its above-mentioned norm is a Banach space.

Example 1.6. (Sobolev Spaces) Let k be a nonnegative integer and let $1 \leq p \leq \infty$. Let $\mathscr{O} \subset \mathbb{R}^n$ be an open subset.

Define the Sobolev space $W^{k,p}(\mathscr{O})$ by

$$W^{k,p}(\mathscr{O}) := \{\phi \in L^p(\mathscr{O}) :\ D^\alpha \phi \in L^p(\mathscr{O}) \ \text{for} \ |\alpha| \leq k\}, \tag{1.8}$$

where $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_i \in \mathbb{N}$ for $i = 1, ..., n$, and $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$. (The derivatives should be understood in the weak sense, that is, the sense of distributions, see, e.g., Adams [4].)

One equips the vector space $W^{k,p}(\mathscr{O})$ with the norm given by

$$\|\varphi\|_{k,p} = \left(\sum_{|\alpha| \leq k} \|D^\alpha \varphi\|_p^p \right)^{\frac{1}{p}} \quad \text{if} \quad 1 \leq p < \infty, \tag{1.9}$$

and

$$\|\varphi\|_{k,\infty} = \max_{|\alpha| \leq k} |D^\alpha \varphi|_\infty \quad \text{if} \quad p = \infty. \tag{1.10}$$

For $p = 2$, the corresponding Sobolev space $W^{k,2}(\mathscr{O})$ is denoted by $H^k(\mathscr{O})$.

If $C_0^\infty(\mathscr{O})$ denotes the collection of functions of class C^∞ with compact support in $\mathscr{O}$, we then define $W_0^{k,p}(\mathscr{O})$ to be the closure of $C_0^\infty(\mathscr{O})$ in the space $W^{k,p}(\mathscr{O})$. Here again, when $p = 2$, the space $W_0^{k,2}(\mathscr{O})$ is denoted by $H_0^k(\mathscr{O})$. Notice that if $\mathscr{O} = \mathbb{R}^n$, then $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$.

Theorem 1.3. *The Sobolev space $W^{k,p}(\mathscr{O})$ equipped with the norm $\|\cdot\|_{k,p}$ is a Banach space.*

To prove Theorem 1.3 we need the following technical lemma whose proof is quite straightforward.

Lemma 1.3. *Let $(u_n)_{n\in\mathbb{N}} \in L^1_{loc}(\mathscr{O})$ such that $u_n \to u$ as $n \to \infty$ in $L^1_{loc}(\mathscr{O})$. Suppose $D^\alpha u_n \in L^1_{loc}(\mathscr{O})$ and $D^\alpha u_n \to v$ in $L^1_{loc}(\mathscr{O})$. Then $D^\alpha u = v$.*

We are now ready to prove Theorem 1.3.

Proof. (Theorem 1.3) Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{k,p}(\mathscr{O})$. Clearly, all the sequences $(D^\alpha u_n)_{n\in\mathbb{N}}$ for $|\alpha| \leq k$ are Cauchy sequences in $L^p(\mathscr{O})$. Now since $L^p(\mathscr{O})$ is a Banach space, then there exist two functions u and u_α which belong to $L^p(\mathscr{O})$ such that

$$\|u_n - u\|_p \to 0 \ \text{ and } \ \|D^\alpha u_n - u_\alpha\|_p \to 0 \ \text{ as } \ n \to \infty.$$

In view of the above, we also have $u_n \to u$, $D^\alpha u_n \to u_\alpha$ in $L^1_{loc}(\mathscr{O})$ as $n \to \infty$. Now, using Lemma 1.3, it follows that $D^\alpha u = u_\alpha$.

For more on these spaces and related issues, we refer the reader to the excellent book by Adams [4].

Example 1.7. (Banach Space of Bounded Continuous Functions) Let $(\mathscr{B}, \|\cdot\|)$ be a Banach space over $\mathbb{F}$ and let $BC(\mathbb{R}, \mathscr{B})$ be the space of all bounded continuous functions equipped with sup norm introduced in Example 1.1.

Theorem 1.4. *The space $BC(\mathbb{R}, \mathscr{B})$ equipped with the sup norm $\|\cdot\|_\infty$ given above is a Banach space.*

In order to prove Theorem 1.4, we will make use of the fact that $L^\infty(\mathbb{R}, \mathscr{B})$ is a Banach space (see Remark 1.2).

Proof. Using the fact $BC(\mathbb{R}, \mathscr{B})$ is a subspace of the Banach space $L^\infty(\mathbb{R}, \mathscr{B})$, it is sufficient to prove that $BC(\mathbb{R}, \mathscr{B})$ is closed in $L^\infty(\mathbb{R}, \mathscr{B})$. Indeed, let $(f_n)_{n\in\mathbb{N}} \subset BC(\mathbb{R}, \mathscr{B})$ be a sequence of functions such that

$$\|f_n - f\|_\infty \mapsto 0 \ \text{ as } \ n \mapsto \infty,$$

where $f \in L^\infty(\mathbb{R}, \mathscr{B})$.

To complete the proof we have to prove that $f \in BC(\mathbb{R}, \mathscr{B})$. First of all, it is clear that f is bounded. Now, let $x_0 \in \mathbb{R}$ (fixed) and let $\varepsilon > 0$. Choose n such that

$$\|f_n - f\|_\infty < \frac{\varepsilon}{3}.$$

Since the sequence of functions $(f_n)_{n\in\mathbb{N}}$ is continuous, then there exists a neighborhood W_{x_0} of x_0 such that

$$\|f_n(x) - f_n(x_0)\| < \frac{\varepsilon}{3}$$

for all $x \in W_{x_0}$.

Clearly, if $x \in W_{x_0}$, then

$$\begin{aligned}\|f(x) - f(x_0)\| &\leq \|f_n(x) - f(x)\| + \|f_n(x) - f_n(x_0)\| \\ &+ \|f_n(x_0) - f(x_0)\| \\ &\leq \|f_n(x) - f_n(x_0)\| + 2 \sup_{y \in W_{x_0}} \|f_n(y) - f(y)\| \\ &< 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon.\end{aligned}$$

1.1.4 Hölder and Lipschitz Spaces

Let $J \subset \mathbb{R}$ be a subset (possibly unbounded). Let $C^m(J, \mathscr{B})$ ($m \in \mathbb{N}$) be the space of m-times continuously differentiable functions from J into $\mathscr{B}$.

Define

$$BC^m(J, \mathscr{B}) = \{f \in C^m(J, \mathscr{B}) : f^{(k)} \in BC(J, \mathscr{B}), \ \ k = 0, 1, \ldots, m\}$$

equipped with the norm

$$\|f\|_{BC^m(J,\mathscr{B})} := \sum_{k=0}^{m} \|f^{(k)}\|_\infty, \quad \forall f \in BC^m(J, \mathscr{B}).$$

The Banach spaces of Hölder spaces of continuous functions $C^\alpha(J, \mathscr{B})$ and $C^{\alpha+k}(J, \mathscr{B})$ for $\alpha \in (0,1)$ and $k \in \mathbb{N}$ are respectively defined by

$$C^{\alpha}(J,\mathscr{B}) = \{f \in BC(J,\mathscr{B}):\ [f]_{C^{\alpha}(J,\mathscr{B})} = \sup_{t,s\in\mathbb{R},s<t} \frac{\|f(t)-f(s)\|}{(t-s)^{\alpha}} < \infty\}$$

equipped with the norm

$$\|f\|_{C^{\alpha}(J,\mathscr{B})} = \|f\|_{\infty} + [f]_{C^{\alpha}(J,\mathscr{B})}, \text{ and}$$

$$C^{\alpha+k}(J,\mathscr{B}) = \{f \in BC^{k}(J,\mathscr{B}) : f^{(k)} \in C^{\alpha}(J,\mathscr{B})\}$$

equipped with the norm

$$\|f\|_{C^{k+\alpha}(J,\mathscr{B})} = \|f\|_{BC^{k}(J,\mathscr{B})} + [f^{(k)}]_{C^{\alpha}(J,\mathscr{B})}.$$

Proposition 1.5. *The Hölder spaces $C^{\alpha}(J,\mathscr{B})$ and $C^{\alpha+k}(J,\mathscr{B})$ for $\alpha \in (0,1)$ and $k \in \mathbb{N}$ equipped with their corresponding norms are respectively Banach spaces.*

Similarly, the Lipschitz space $Lip(J,\mathscr{B})$ is defined by

$$\mathrm{Lip}(J,\mathscr{B}) = \{f \in BC(J,\mathscr{B}):\ [f]_{Lip(J,\mathscr{B})} = \sup_{t,s\in\mathbb{R},s<t} \frac{\|f(t)-f(s)\|}{(t-s)} < \infty\},$$

and is equipped with the norm defined by

$$\|f\|_{\tilde{Lip}(J,\mathscr{B})} = \|f\|_{\infty} + [f]_{Lip(J,\mathscr{B})}.$$

Let $\mathscr{O} \subset \mathbb{R}^{N}$ be an open bounded subset and let $\alpha \in (0,1)$. As above, we define the Hölder space $C_{b}^{\alpha}(\overline{\mathscr{O}})$ (when $\mathscr{O}$ is bounded, we drop the subscript b) as the collection of all bounded continuous functions $u : \overline{\mathscr{O}} \mapsto \mathbb{C}$ such that

$$[u]_{C_{b}^{\alpha}(\overline{\mathscr{O}})} := \sup_{x\neq y\in\overline{\mathscr{O}}} \frac{|u(x)-u(y)|}{\|x-y\|^{\alpha}} < \infty.$$

The space $C_{b}^{\alpha}(\overline{\mathscr{O}})$ is a Banach space when it is equipped with the norm defined by

$$\|u\|_{C_{b}^{\alpha}(\overline{\mathscr{O}})} = \|u\|_{\infty} + [u]_{C_{b}^{\alpha}(\overline{\mathscr{O}})}$$

for all $u \in C_{b}^{\alpha}(\overline{\mathscr{O}})$.

Similarly, if $k \in \mathbb{N}$, we define $C_{b}^{k+\alpha}(\overline{\mathscr{O}})$ as the collection of all differentiable functions up to the order k in $\overline{\mathscr{O}}$ with bounded derivatives such that $D^{\beta} \in C_{b}^{\alpha}(\overline{\mathscr{O}})$ for any multiindex β with $|\beta| = k$. Clearly, $C_{b}^{k+\alpha}(\overline{\mathscr{O}})$ is a Banach space when it is equipped with the norm defined by

$$\|u\|_{C_{b}^{k+\alpha}(\overline{\mathscr{O}})} = \sum_{|\beta|\leq k} \|D^{\beta}u\|_{\infty} + \sum_{|\beta|=k} [D^{\beta}]_{C_{b}^{\alpha}(\overline{\mathscr{O}})}.$$

For more on these spaces, we refer the reader to Lunardi et al. [131].

1.2 Hilbert Spaces

Hilbert spaces play a crucial role in mathematics as well as in fields such as engineering, physics, and quantum mechanics. Hilbert spaces are generalizations of Euclidean spaces. Key properties of those spaces will be introduced and discussed in this section.

1.2.1 Basic Definitions

Definition 1.7. Let $\mathscr{H}$ be a vector space over the field $\mathbb{F}$. An *inner product* or *scalar product* on $\mathscr{H}$ is a mapping, $\langle \cdot, \cdot \rangle : \mathscr{H} \times \mathscr{H} \mapsto \mathbb{C}, \ (x,y) \mapsto \langle x,y \rangle$ satisfying $\forall x,y,z \in \mathscr{H}$, and $\lambda \in \mathbb{F}$,

(i) $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$;
(ii) $\langle \lambda x,y \rangle = \lambda \langle x,y \rangle$; and $\langle x,\lambda y \rangle = \overline{\lambda} \langle x,y \rangle$;
(iii) $\langle x,y \rangle = \overline{\langle y,x \rangle}$; and
(iv) $\langle x,x \rangle \geq 0$; and $\langle x,x \rangle = 0$ if and only if $x = 0$.

Definition 1.8. A vector space $\mathscr{H}$ over $\mathbb{F}$ endowed with an inner product is called a *pre–Hilbert space*.

If $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ is a pre–Hilbert space, then we define a norm $\|\cdot\|$ on $\mathscr{H}$ by setting $\|x\| := \sqrt{\langle x,x \rangle}$ for all $x \in \mathscr{H}$. In this event, we say that the norm $\|\cdot\|$ is generated by the inner product $\langle \cdot, \cdot \rangle$.

Example 1.8. Let $\tilde{\mathscr{H}}$ be the vector of all continuous functions $u : \mathbb{R} \mapsto \mathbb{C}$ such that

$$N(u) := \left(\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} |u(t)|^2 dt \right)^{1/2} < \infty.$$

Since there exist nonzero continuous functions on $\mathbb{R}$ for which $N(u) = 0$ it follows that N is not a norm on $\tilde{\mathscr{H}}$. Now let $Ker(N) = \{u \in \tilde{\mathscr{H}} : N(u) = 0\}$. It is clear that N is a norm on the quotient space $\tilde{\mathscr{H}}/Ker(N)$. Let $\mathscr{H}$ denote the completion of the quotient normed vector space $\tilde{\mathscr{H}}/Ker(N)$. The corresponding inner product on $\mathscr{H}$ is defined by

$$\langle u,v \rangle := \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} u(t)\overline{v(t)} dt$$

for all $u,v \in \mathscr{H}$.

Clearly, $\mathscr{H}$ equipped of the inner product $\langle \cdot, \cdot \rangle$ defined above is a pre–Hilbert space.

Example 1.9. Let $\mathscr{O} \subset \mathbb{C}$ be an open bounded subset and let $A^2(\mathscr{O})$ be the vector space of all holomorphic functions $u : \mathscr{O} \mapsto \mathbb{C}$ such that

$$\|u\| := \left(\int_{\mathscr{O}} |u(z)|^2 dxdy \right)^{1/2} < \infty, \quad (z = x + iy).$$

Define the inner product

$$\langle u, v \rangle := \int_{\mathscr{O}} u(z)\overline{v(z)} dxdy$$

for all $u, v \in A^2(\mathscr{O})$.

Then, $A^2(\mathscr{O})$ equipped with the above-mentioned inner product is a pre–Hilbert space.

Theorem 1.5. (Parallelogram Law) *In a pre–Hilbert space $\mathscr{H}$, we have*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \quad \forall x, y \in \mathscr{H}.$$

Proof. Clearly, $\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\,\langle x, y \rangle$. Similarly, replacing y by $-y$ in the previous identity, we obtain

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\mathrm{Re}\,\langle x, y \rangle,$$

and hence

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= 2\left(\|x\|^2 + \|y\|^2\right) + 2\mathrm{Re}\,\langle x, y \rangle - 2\mathrm{Re}\,\langle x, y \rangle \\ &= 2\left(\|x\|^2 + \|y\|^2\right) \end{aligned}$$

for all $x, y \in \mathscr{H}$.

Theorem 1.6. (Cauchy–Schwarz Inequality) *In a pre–Hilbert space $\mathscr{H}$, we have*

$$|\langle x, y \rangle| \leq \|x\| \,.\, \|y\|, \quad \forall x, y \in \mathscr{H}. \tag{1.11}$$

The equality holds if and only if x and y are linearly dependent.

Proof. Note that

$$0 \leq \langle x - \lambda y, x - \lambda y \rangle = \langle x, x \rangle - 2\mathrm{Re}\,[\lambda \langle y, x \rangle] + |\lambda|^2 \langle y, y \rangle.$$

Suppose without lost of generality that $\langle x, y \rangle \neq 0$. Now, taking $\lambda = \dfrac{\langle x, x \rangle}{\langle y, x \rangle}$ it easily follows that

$$0 \leq -\langle x, x \rangle^2 + \frac{\langle x, x \rangle^4 \langle y, y \rangle^2}{|\langle y, x \rangle|^2},$$

which gives the Cauchy–Schwarz Inequality. In addition to the above, one can easily see that $|\langle x, y \rangle| = \langle x, x \rangle \langle y, y \rangle$ if and only if $x = \lambda y$.

Theorem 1.7. (Polarization Identity) *In a pre–Hilbert space $\mathscr{H}$,*

$$\langle x,y\rangle = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2\right\},$$

for all $x,y\in\mathscr{H}$.

Definition 1.9. A pre–Hilbert space $\mathscr{H}$ which is complete is called a Hilbert space.

Example 1.10. (The Space $\mathbb{C}^d$) The space $\mathbb{C}^d$, that is, the d–dimensional space of complex numbers, equipped with the inner product and norm defined by: for all $u=(u_1,\dots,u_n),\ v=(v_1,v_2,\dots,v_n)\in\mathbb{C}^n$,

$$\langle u,v\rangle = \sum_{k=1}^{n} u_k\bar{v_k} \ \text{ and } \ \|u\| = \left(\sum_{k=1}^{n}|u_k|^2\right)^{1/2}$$

is a Hilbert space.

Example 1.11. ($L^2(\mathscr{O})$–Space) The space $L^2(\mathscr{O})$ (see Example 1.5) equipped with its natural norm $\|\cdot\|_2$ and the inner product defined for all $u,v\in L^2(\mathscr{O})$ by

$$\langle u,v\rangle := \int_{\mathscr{O}} u(t)\overline{v(t)}dt$$

is a Hilbert space.

Example 1.12. (The Sobolev Space $H^k(\mathscr{O})$) The Sobolev space $H^k(\mathscr{O})$ previously defined is a Hilbert space when equipped with the inner product

$$\langle u,v\rangle_{H^k(\mathscr{O})} := \int_{\mathscr{O}} \sum_{|\alpha|\le k} D^\alpha u(x)\overline{D^\alpha v(x)}dx \ \text{ for all } \ u,v\in H^k(\mathscr{O}).$$

1.2.2 Orthogonality

Definition 1.10. Let $(\mathscr{H},\langle\cdot,\cdot\rangle,\|\cdot\|)$ be a Hilbert space over the field $\mathbb{F}$. If u and v are two vectors in $\mathscr{H}$, then one says u and v are orthogonal and write $u\perp v$ if $\langle u,v\rangle = 0$.

An immediate consequence of the orthogonality is the so-called *Pythagorean* theorem, which says that if $u\perp v$, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

To see it, it suffices to use the identity

$$\|u+v\|^2 = \|u\|^2 + 2\Re e\langle u,v\rangle + \|u\|^2$$

for all $u, v \in \mathscr{H}$.

A system of vectors $(v_n)_{n\in\mathbb{N}}$ is said to be an orthogonal system whenever $v_n \perp v_m$ for all $n \neq m$. If in addition, $\|v_n\| = 1$, then the system is said to be orthonormal.

Example 1.13. In $l^2(\mathscr{B})$, consider the vectors $e_0 = (1,0,...)$, $e_1 = (0,1,0,...)$, ..., $e_n = (0,0,0,...,1,0,0,...)$ where the 1 appears in the $(n+1)$-th position. It is then clear that $(e_n)_{n\in\mathbb{N}}$ is an orthonormal sequence of $l^2(\mathscr{B})$.

Example 1.14. In $L^2[-\pi,\pi]$ equipped with its natural inner product and norm given by

$$\langle u, v\rangle = \int_{-\pi}^{\pi} u(t)\bar{v}(t)dt \text{ and } \|u\|_2 = \left(\int_{-\pi}^{\pi} |u(t)|^2\right)^{1/2},$$

one can easily check that the system of vectors $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ form an orthonormal system.

Definition 1.11. A system of vectors $(e_n)_{n\in\mathbb{N}}$ is said to be a basis of $\mathscr{H}$ if each $u \in \mathscr{H}$ there exists a unique sequence (u_n) with $u_i \in \mathbb{F}$ such that

$$u = \sum_{n\in\mathbb{N}} u_i e_i.$$

If in addition $(e_n)_{n\in\mathbb{N}}$ is an orthonormal system of vectors, then it is said to be an orthonormal base for $\mathscr{H}$.

Classical examples of orthonormal bases include those two examples mentioned above.

Definition 1.12. Let $\mathscr{V}$ be a subspace of a Hilbert space $\mathscr{H}$. The orthogonal complement $\mathscr{V}^\perp$ of $\mathscr{V}$ is defined by

$$\mathscr{V}^\perp = \left\{u \in \mathscr{H} : \ \langle u, v\rangle = 0, \ \forall v \in \mathscr{V}\right\}.$$

Note that the orthogonal complement $\mathscr{V}^\perp$ of a closed subspace $\mathscr{V}$ of $\mathscr{H}$ is very often denoted by $\mathscr{H} \ominus \mathscr{V}$ in the literature.

Proposition 1.6. *Let $(e_k)_k$ for $k = 0,1,2,...,n$ be a finite orthogonal system of vectors in the Hilbert space $\mathscr{H}$. Then the following holds:*

$$\sum_{k=0}^{n} |\langle v, e_k\rangle|^2 \leq \|v\|^2, \ \forall v \in \mathscr{H}.$$

Proof. First of all, write $v = \sum_{k=1}^{n} \langle v, e_k\rangle e_k + \left(v - \sum_{k=1}^{n} \langle v, e_k\rangle e_k\right)$. Using the fact that

$$\left\langle \sum_{k=1}^{n} \langle v, e_k\rangle . e_k, v - \sum_{k=1}^{n} \langle v, e_k\rangle . e_k \right\rangle = 0$$

it follows that

$$\|v - \sum_{k=1}^{n} \langle v, e_k \rangle \,.\, e_k\|^2 + \| \sum_{k=1}^{n} \langle v, e_k \rangle \,.\, e_k\| = \|v\|^2,$$

and hence

$$\|v\|^2 \geq \| \sum_{k=1}^{n} \langle v, e_k \rangle \,.\, e_k\|^2.$$

Now it can be easily shown that

$$\| \sum_{k=1}^{n} \langle v, e_k \rangle \,.\, e_k\|^2 = \sum_{k=1}^{n} |\langle v, e_k \rangle|^2,$$

which completes the proof.

A straightforward consequence of Proposition 1.6 is the next corollary whose proof will be omitted.

Corollary 1.1. (Bessel's Inequality) *Let $(e_k)_{k\in\mathbb{N}}$ be an orthogonal system of vectors in the Hilbert space $\mathbb{H}$. Then the following holds:*

$$\sum_{k=0}^{\infty} |\langle u, e_k \rangle|^2 \leq \|u\|^2, \ \forall u \in \mathscr{H}.$$

A system $(f_n)_{n\in\mathbb{N}}$ in a normed vector space $\mathscr{B}$ is said to be a complete system if the vector space spanned by $(f_n)_{n\in\mathbb{N}}$ is a dense set in $\mathscr{B}$.

Definition 1.13. A normed vector space $(\mathscr{B}, \|\cdot\|)$ is said to be separable if it contains a dense countable subset.

Theorem 1.8. *A Hilbert space $\mathscr{H}$ is separable if and only if it contains a complete orthonormal system $(e_n)_{n\in\mathbb{N}}$.*

For the proof of Theorem 1.8 we refer the reader to any good book in functional analysis, in particular to [69].

Example 1.15. Consider the Hilbert space given in Example 1.8. It can be shown that $\mathscr{H}$ contains an orthonormal system given by $(e^{i\alpha t})_{\alpha\in\mathbb{R}}$. And since the previous orthonormal system is uncountable it follows that $\mathscr{H}$ is not separable.

Example 1.16. $l^2(\mathscr{B})$ and $L^2[-\pi, \pi]$ considered respectively in Examples 1.13 and 1.14 are separable.

1.2.3 Projections

A projection P on a (normed) vector space $\mathscr{H}$ is a linear operator (see Section 2 for the definition of the linearity) satisfying $P^2 = P$. If P is a projection a (normed) vector space $\mathscr{H}$ and if $N(P)$ and $R(P)$ stand respectively for its null space and range, then one can easily check that the following properties hold:

(i) $R(P) = N(I-P) = \{u \in \mathscr{H} : Pu = u\}$;

(ii) $N(P) = R(I-P)$;

(iii) $R(P) \cap N(P) = \{0\}$; and

(iv) $\mathscr{H} = N(P) + R(P)$. Moreover, if P is continuous, then $\mathscr{H} = N(P) \oplus R(P)$.

Definition 1.14. Let $\mathscr{H}$ be a Hilbert space. A subset $C \subset \mathscr{H}$ is said to be convex if, for all $x, y \in C$ and all $t \in [0,1]$, the point $(1-t)x + ty \in C$.

Theorem 1.9. *If C is a closed convex subset of $\mathscr{H}$, and x a point in $\mathscr{H}$, then there exists a unique $y \in C$ such that*

$$\|y - x\| = \inf_{z \in C} \|z - x\|.$$

The unique $y \in C$ satisfying $\|y - x\| = \inf_{z \in C} \|z - x\|$ is then called the *projection* of x onto C.

Proof. We reproduce here a proof, which was given in Diagana [61]. Set $\gamma = \inf_{z \in C} \|x - z\|$. Obviously, there exists a minimizing sequence $(z_n)_{n \in \mathbb{N}} \subset C$ such that $\|z_n\| \to \gamma$ as $n \to \infty$. Now since C is convex, it is clear that $\frac{1}{2}(z_n + z_m) \in C$ for all $n, m \in \mathbb{N}$. And hence

$$\gamma \leq \frac{1}{2} \|z_n + z_m\| \tag{1.12}$$

for all $n, m \in \mathbb{N}$.

Using the Parallelogram identity, one can easily see that

$$\|z_n - z_m\|^2 = 2\|x = z_n\|^2 + 2\|x - z_m\|^2 - 4\left\|x - \frac{1}{2}(z_n + z_m)\right\|^2 \tag{1.13}$$

for all $n, m \in \mathbb{N}$.

Since $\|x - z_n\| \to \gamma$ and $\|x - z_m\| \to \gamma$ as $n, m \to \infty$, then (1.13) implies by using (1.12) that

$$\|z_n - z_m\|^2 \leq 2\|x - z_n\|^2 + 2\|x - z_m\|^2 - 4\gamma^2 \to 0$$

as $n, m \to \infty$.

Consequently, $(z_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and hence there exists $y \in \mathscr{H}$ such that $z_n \to y$. Now since C is closed, it is clear that $y \in C$. Obviously, $\gamma = \|x - y\|$.

Now suppose that there exists another $\tilde{y} \in C$ such that $\gamma = \|x - \tilde{y}\|$. Clearly,

$$\|y - \tilde{y}\|^2 = 4\gamma^2 - 4\left\|x - \frac{1}{2}(y + \tilde{y})\right\|^2 \leq 0,$$

as $\frac{1}{2}(y + \tilde{y}) \in C$. Therefore, $y = \tilde{y}$, which shows that the element y such that $\gamma = \|x - y\|$ is unique.

Example 1.17. This example was first given in [69] as a practice problem and then discussed in [61]. Let $a > 0$. Define the subspaces $\mathscr{H}_{odd}$ and $\mathscr{H}_{even}$ of $L^2[-a,a]$ as follows:

$$\mathscr{H}_{odd} = \{f \in L^2[-a,a] : f(-t) = -f(t)\}$$

and

$$\mathscr{H}_{even} = \{f \in L^2[-a,a] : f(-t) = f(t)\}.$$

Obviously, both $\mathscr{H}_{odd}$ and $\mathscr{H}_{even}$ are infinite-dimensional vector spaces. Moreover, if $f \in \mathscr{H}_{odd}$ and $g \in \mathscr{H}_{even}$, then

$$\langle f, g\rangle = \int_{-r}^{r} f(t)\overline{g(t)}dt = 0,$$

as the function $t \mapsto f(t)\overline{g(t)}$ is odd, and hence $\mathscr{H}_{odd} \perp \mathscr{H}_{even}$. Furthermore, each $f \in L^2[-r,r]$ can be uniquely decomposed as

$$f = f_{even} + f_{odd}$$

where

$$f_{even}(t) = \frac{f(t)+f(-t)}{2}$$

and

$$f_{odd}(t) = \frac{f(t)-f(-t)}{2}.$$

Consequently, $L^2[-r,r] = \mathscr{H}_{odd} \oplus \mathscr{H}_{even}$. Therefore, $\mathscr{H}_{odd}$ is the orthogonal complement $\mathscr{H}_{even}$ and $\mathscr{H}_{even}$ is the orthogonal complement of $\mathscr{H}_{odd}$ ($\mathscr{H}_{odd} = L^2[-a,a] \ominus \mathscr{H}_{even}$), hence both $\mathscr{H}_{odd}$ and $\mathscr{H}_{even}$ are closed.

The projection of each $f \in L^2[-a,a]$ onto $\mathscr{H}_{odd}$, that is,

$$P_{\mathscr{H}_{odd}} = f_{odd}(t) = \frac{f(t)-f(-t)}{2}.$$

Similarly, its projection onto $\mathscr{H}_{even}$, that is,

$$P_{\mathscr{H}_{even}} = f_{even}(t) = \frac{f(t)+f(-t)}{2}.$$

Furthermore, $d(f, \mathscr{H}_{odd}) = \|f_{even}\|_2$ and $d(f, \mathscr{H}_{even}) = \|f_{odd}\|_2$.

For instance, if $f(t) = t^2 + t$, then

$$d(t^2+t, \mathscr{H}_{even}) = \|t\|_2 = \left(\int_{-r}^{r} t^2 dt\right)^{1/2} = \sqrt{\frac{2r^3}{3}}$$

and

$$d(t^2+t, \mathscr{H}_{odd}) = \|t^2\|_2 = \left(\int_{-r}^{r} t^4 dt\right)^{1/2} = \sqrt{\frac{2r^5}{5}}.$$

Similarly, for $f(t) = \cos t + \sin t$, we have

$$d(\cos t+\sin t,\mathscr{H}_{even}) = \|\sin t\|_2 = \left(\int_{-r}^{r}(\sin t)^2dt\right)^{1/2} = \sqrt{r-\frac{\sin(2r)}{2}}$$

and

$$d(\cos t+\sin t,\mathscr{H}_{odd}) = \|(\cos t)^2\|_2 = \left(\int_{-r}^{r}(\cos t)^2dt\right)^{1/2} = \sqrt{r+\frac{\sin 2r}{2}}.$$

Theorem 1.10. *Let $M \subset \mathscr{H}$ be a closed subspace, then $x-P_Mx$ is orthogonal to M, that is, $\langle x-P_Mx, y\rangle = 0, \quad \forall y \in M$.*

Proof. We reproduce here a proof, which was given in Diagana [61]. Let $z \in \mathbb{C}$ and let $y \in M$. Since M is subspace of $\mathscr{H}$ and $P_Mx \in M$ it follows that $P_Mx+zy \in M$. Now, let $\gamma = \|x-P_Mx\|$. From Theorem 1.9 it follows that

$$\begin{aligned}\gamma^2 &\leq \|x-(P_Mx+zy)\|^2\\ &= \|x-P_Mx\|^2+|z|^2\|y\|^2-\bar{z}\langle x-P_Mx,y\rangle - z\langle y,x-P_Mx\rangle,\end{aligned}$$

and hence

$$0 \leq |z|^2\|y\|^2-\bar{z}\langle x-P_Mx,y\rangle - z\langle y,x-P_Mx\rangle.$$

As z is arbitrary, then taking $z = z'\langle x-P_Mx,y\rangle$, where $z' \in \mathbb{R}$ it easily follows that

$$0 \leq z'^2|\langle x-P_Mx,y\rangle|^2\|y\|^2 - 2z'|\langle x-P_Mx,y\rangle|^2.$$

Now since the last inequality holds for each $z' \in \mathbb{R}$ it follows that $|\langle x-P_Mx,y\rangle|^2 = 0$, that is, $\langle x-P_Mx,y\rangle = 0$. Finally, since $\langle x-P_Mx,y\rangle = 0$ for any $y \in M$, it then follows that $x-P_Mx$ is orthogonal to M.

Theorem 1.11. *(Orthogonal Decomposition) Let M be a closed subspace of a Hilbert space $\mathscr{H}$. Every $x \in \mathscr{H}$ has a unique decomposition given by*

$$x = y+z, \quad \textit{where} \quad y = P_Mx \quad \textit{and} \quad z = (I-P_M)x. \tag{1.14}$$

Proof. We reproduce here a proof, which was given in Diagana [61]. Suppose $x = y+z$ where $y \in M$ and $z \in M^{\perp}$. Now one can write $x = P_Mx+(x-P_Mx)$ where $P_Mx \in M$ and $x-P_Mx \in M^{\perp}$ (see Theorem 1.10). Obviously,

$$y-P_Mx = (x-P_Mx)-z \quad \text{and} \quad y-P_Mx \perp (x-P_Mx)-z.$$

Hence, $x-P_Mx = 0 = (x-P_Mx)-z$, as a vector orthogonal to itself.

1.3 Bibliographical Notes

The classical results on Banach and Hilbert spaces and some of their proofs found in the text are mainly taken from Diagana [51], Eidelman, Milman, and Tsolomitis [69], Gohberg, Goldberg, and Kaashoek [78], Kato [105], Lax [115], Naylor and Sell [146], Rudin [159], Weidmann [176], and Yosida [186]. The elementary properties of orthogonal projections given in the text follow essentially the presentation in Locker [130] and Conway [40].

Chapter 2
Bounded and Unbounded Linear Operators

In this chapter, unless otherwise mentioned, $(\mathscr{B}, \|\cdot\|)$ and $(\mathscr{B}', \|\cdot\|')$ stand for Banach spaces over the same field $\mathbb{F}$. Similarly, $\mathscr{H}$ will denote a Hilbert space equipped with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$. Further, I and O stand respectively for the identity and zero operators of $\mathscr{B}$ defined by $Ix = x$ and $Ox = O$ for all $x \in \mathscr{B}$.

2.1 Introduction

This chapter is devoted to the basic material on operator theory, semigroups, evolution families, interpolation spaces, intermediate spaces, and their basic properties needed in the sequel. In each section, illustrative examples will be discussed in–depth. The technical Lemma 2.2 (Diagana et al. [52]) and Lemma 2.4 (Diagana [62]) will play a key role throughout the book. Detailed proofs of these lemmas will be discussed at the very end of this chapter.

2.2 Linear Operators

2.2.1 Bounded Operators

A linear operator $A : \mathscr{B} \to \mathscr{B}'$ is a transformation which maps linearly $\mathscr{B}$ in $\mathscr{B}'$, that is, $A(\alpha u + \beta v) = \alpha Au + \beta Av$ for all $u, v \in \mathscr{B}$ and $\alpha, \beta \in \mathbb{F}$.

Definition 2.1. A linear operator $A : \mathscr{B} \to \mathscr{B}'$ is said to be bounded if there exists $K \geq 0$ such that

$$\|Au\|' \leq K \|u\| \quad \text{for each} \quad u \in \mathscr{B}. \tag{2.1}$$

P.H. Bezandry and T. Diagana, *Almost Periodic Stochastic Processes*,
DOI 10.1007/978-1-4419-9476-9_2,

If $A : \mathscr{B} \to \mathscr{B}'$ is a bounded linear operator, then its norm $\|A\|$ is the smallest K for which (2.1) holds, that is,

$$\|A\| := \sup_{u \neq 0} \frac{\|Au\|'}{\|u\|}. \tag{2.2}$$

The collection of all bounded linear operators from $\mathscr{B}$ into $\mathscr{B}'$ is denoted by $B(\mathscr{B}, \mathscr{B}')$. In particular, $B(\mathscr{B}, \mathscr{B})$ is denoted by $B(\mathscr{B})$. It can be shown that $B(\mathscr{B}, \mathscr{B}')$ equipped with the operator topology given above is a Banach space.

Example 2.1. Let $\mathscr{B} = C[0,1]$ be the collection of all continuous functions from $[0,1]$ in the complex plan $\mathbb{C}$ equipped with its corresponding sup norm defined for each function $f \in C[0,1]$ by

$$\|f\|_\infty := \max_{t \in [0,1]} |f(t)|.$$

Define the integral operator A by setting for each $f \in C[0,1]$,

$$Af = \int_0^1 K(t,\tau) f(\tau) d\tau$$

where K is a jointly continuous function.

Clearly, the operator A is linear. For the continuity, it suffices to see that

$$\|Af\|_\infty \leq \|f\|_\infty \max_{t \in [0,1]} \Big(\int_0^1 |K(t,\tau)| d\tau \Big).$$

It can be shown that (see for instance [79])

$$\|A\| = \max_{t \in [0,1]} \Big(\int_0^1 |K(t,\tau)| d\tau \Big).$$

Example 2.2. Let $\mathscr{B} = BC([0,\infty), \mathbb{C})$ be the collection of all bounded continuous functions from $[0,\infty)$ in $\mathbb{C}$ equipped with its corresponding supnorm $\|\cdot\|_\infty$. Define the transformation

$$(Af)(t) = \frac{1}{t} \int_0^t f(s) ds.$$

Using the L'Hôpital rule, it can be easily seen that $\lim_{t \to 0} (Af)(t) = f(0)$. Clearly, A is linear. Moreover $\|Af\|_\infty \leq \|f\|_\infty$, that is, A is a bounded linear operator.

Theorem 2.1. *If $A : \mathscr{B} \to \mathscr{B}'$ is a linear operator, then the following statements are equivalent:*

(i) A is continuous;

(ii) A is continuous at 0;

(iii) there exists $K > 0$ such that $\|Au\|' \leq K \,.\, \|u\|$ for each $u \in \mathscr{B}$.

Proof. Clearly, (i) yields (ii). Suppose (ii) holds. Hence there exists $\eta > 0$ such that $\|Ax\|' \leq 1$ whenever $\|x\| \leq \eta$. Now for each nonzero $x \in \mathscr{B}$,

$$\left\| \frac{\eta x}{\|x\|} \right\| = \eta.$$

Now

$$1 \geq \left\| A\left(\frac{\eta x}{\|x\|}\right) \right\|' = \frac{\eta \|Ax\|'}{\|x\|},$$

and hence $\|Ax\|' \leq \eta^{-1}\|x\|$ and (iii) holds.

Now if (iii) holds, it is then clear that

$$\|Ax - Ax_0\|' = \|A(x - x_0)\|' \leq K\|x - x_0\|.$$

Consequently, for each $\varepsilon > 0$ there exists $\eta = \dfrac{\varepsilon}{K}$ such that $\|Ax - Ax_0\|' < \varepsilon$ whenever $\|x - x_0\| \leq \eta$. Therefore, A is continuous at x_0. Since $x_0 \in \mathscr{B}$ was arbitrary, it follows that A is continuous everywhere in $\mathscr{B}$.

Proposition 2.1. *If A, B are bounded linear operators on $\mathscr{B}$ and if $\lambda \in \mathbb{C}$, then $A+B$, λA, and AB are also bounded operators. Moreover,*

(i) $\|A+B\| \leq \|A\| + \|B\|$;
(ii) $\|\lambda A\| = |\lambda| . \|A\|$;
(iii) $\|AB\| \leq \|A\| \, \|B\|$.

Proof. (i) We make use of the inequality $\|Ax\| \leq \|A\|\|x\|$ for each $x \in \mathscr{B}$, which can be easily deduced from the definition of the norm $\|A\|$ of A. Let $x \neq 0$. We have

$$\begin{aligned} \|(A+B)x\| &\leq \|Ax\| + \|Bx\| \\ &\leq \|A\|\|x\| + \|B\|\|x\|, \end{aligned}$$

and hence

$$\frac{\|(A+B)x\|}{\|x\|} \leq \|A\| + \|B\|,$$

and therefore,

$$\|A+B\| \leq \|A\| + \|B\|.$$

(ii) $\|\lambda A\| = \sup\limits_{0 \neq x \in \mathscr{B}} \dfrac{\|\lambda Ax\|}{\|x\|} = |\lambda| . \sup\limits_{0 \neq x \in \mathscr{B}} \dfrac{\|Ax\|}{\|x\|} = |\lambda| . \|A\|$.

(iii) $\|AB\| = \sup\limits_{0 \neq x \in \mathscr{B}} \dfrac{\|ABx\|}{\|x\|} \leq \|A\| . \sup\limits_{0 \neq x \in \mathscr{B}} \dfrac{\|Bx\|}{\|x\|} = \|A\| . \|B\|$.

2.2.1.1 Adjoint For Bounded Operators

Let $A \in B(\mathscr{H})$. Clearly, the quantity $\langle Ax, y \rangle$ is linear in x, conjugate-linear in y and bounded. Therefore, according to the Riesz representation theorem [159], there exists a unique $A^* \in B(\mathscr{H})$ such that

$$\langle Ax, y\rangle = \langle x, A^*y\rangle$$

for all $x, y \in \mathscr{H}$.

The transformation $y \mapsto A^*y$ is called the adjoint of the linear operator A.

Proposition 2.2. *If $A : \mathscr{H} \to \mathscr{H}$ is a bounded linear operator, then $A^* \in B(\mathscr{H})$. Furthermore, $\|A\| = \|A^*\|$.*

Proof. We first show that A^* is a bounded linear operator. Note that

$$\begin{aligned}
\langle u, A^*(\alpha u + \beta w)\rangle &= \langle Au, \alpha u + \beta w\rangle \\
&= \overline{\alpha}\langle Au, v\rangle + \overline{\beta}\langle Au, w\rangle \\
&= \overline{\alpha}\langle u, A^*v\rangle + \overline{\beta}\langle u, A^*w\rangle \\
&= \langle u, \alpha A^*u + \beta A^*w\rangle
\end{aligned}$$

and hence A^* is linear.

Now

$$\begin{aligned}
\|A^*u\|^2 &= \langle A^*u, A^*u\rangle \\
&= \langle AA^*u, u\rangle \\
&\leq \|AA^*u\| \,.\, \|u\| \\
&\leq \|A\| \,.\, \|A^*u\| \,.\, \|u\|
\end{aligned}$$

and hence $\|A^*u\| \leq \|A\| \,.\, \|u\|$, that is, $\|A^*\| \leq \|A\|$.

Similarly, $\|(A^*)^*\| \leq \|A^*\|$. Now using the fact $(A^*)^* = A$ it follows that $\|A\| \leq \|A^*\|$, which completes the proof.

Corollary 2.1. *If $A : \mathscr{H} \to \mathscr{H}$ is a bounded linear operator, then*

$$\|AA^*\| = \|A^*A\| = \|A^*\|^2 = \|A\|^2 .$$

Proof. Using Propositions 2.2 and 2.1 it follows that $\|A^*A\| \leq \|A^*\| \,.\, \|A\| = \|A\| \,.\, \|A\| = \|A\|^2$. It is also clear that $\|A\|^2 \leq \|A^*A\|$, which completes the proof.

Proposition 2.3. *If A, B are bounded linear operators on $\mathscr{H}$ and if $\lambda \in \mathbb{C}$, then*

$$I^* = I, \tag{2.3}$$

$$O^* = O, \tag{2.4}$$

$$(A + B)^* = A^* + B^*, \tag{2.5}$$

$$(\lambda A)^* = \overline{\lambda}\, A^*, \tag{2.6}$$

$$(AB)^* = A^*\, B^*. \tag{2.7}$$

Example 2.3. Let $\mathscr{H} = L^2([\alpha,\beta])$ and let $A : L^2([\alpha,\beta]) \to L^2([\alpha,\beta])$ be the bounded linear operator defined by

$$A\phi(s) = \int_\alpha^\beta V(s,t)\phi(t)dt, \ \ \forall \phi \in L^2([\alpha,\beta]),$$

where $V : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{C}$ is continuous.

It can be easily shown that the adjoint A^* of A is defined by

$$A^*\psi(s) = \int_\alpha^\beta \overline{V(t,s)}\psi(t)dt, \ \ \forall \psi \in L^2([\alpha,\beta]).$$

Definition 2.2. A bounded linear operator $A : \mathscr{H} \mapsto \mathscr{H}$ is called self-adjoint or symmetric if $A = A^*$.

Example 2.4. Consider the integral operator given in Example 2.3. Assuming that V satisfies $V(s,t) = \overline{V(t,s)}$ for all $t,s \in [\alpha,\beta]$, one can easily see that A is symmetric.

Let $A : \mathscr{H} \mapsto \mathscr{H}$ be a bounded linear selfadjoint operator. Then the following properties hold. Their proofs are left as an exercise for the reader:

(i) $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in \mathscr{H}$.

(ii) $\|A\| = \sup_{x\neq 0} \dfrac{|\langle Ax,x\rangle|}{\|x\|^2}$.

(iii) If $B \in B(\mathscr{B})$ is also self-adjoint and if $AB = BA$, then AB is also self-adjoint.

2.2.1.2 The Inverse Operator

Definition 2.3. An operator $A \in B(\mathscr{B})$ is called invertible if there exists $B \in B(\mathscr{B})$ such that $AB = BA = I$. In that event, the operator B is called the inverse operator of A and denoted by $B = A^{-1}$.

Theorem 2.2. *If $A \in B(\mathscr{B})$ is a linear operator such that $\|A\| < 1$, then the operator $I - A$ is invertible.*

Proof. Note that $(I-A)(I+A^2+\ldots+A^n) = I - A^{n+1}$, and $\|A^{n+1}\| \le \|A\|^{n+1} \mapsto 0$ as $n \mapsto \infty$, since $\|A\| < 1$. Consequently,

$$\lim_{n\to\infty}(I-A)(I+A^2+\ldots+A^n) = I \ \text{ in } \ B(\mathscr{B}).$$

Now since $\|A\| < 1$ and that $B(\mathscr{B})$ is a Banach algebra, $S := \lim_{n\to\infty}(I+A^2+\ldots+A^n)$ does exist, hence $(I-A)S = I$. In fact,

$$(I-A)S = I + (I-A)\left[S - (I + A^2 + \ldots + A^n)\right].$$

On the other hand,

$$\begin{aligned}\|(I-A)\left[S-(I+A^2+\ldots+A^n)\right]\| &\leq \|I-S\| \\ &\quad \cdot\ \|S-(I+A^2+\ldots+A^n)\| \\ &\quad \mapsto 0 \ \text{ as } \ n \mapsto \infty,\end{aligned}$$

hence $(I-A)S = I$.

In summary, $(I-A)$ is invertible and $(I-A)^{-1} = S$, where

$$S = \sum_{k=0}^{\infty} A^k \ \ (A^0 \ \text{ being } \ I).$$

Remark 2.1. Note that if $A, B \in B(\mathscr{B})$ are invertible, so is their composition AB. Moreover, $(AB)^{-1} = B^{-1}A^{-1}$.

Similarly, if $A \in B(\mathscr{B})$ is invertible and if $B \in B(\mathscr{B})$ is such that $\|A-B\| < \frac{1}{\|A^{-1}\|}$, then B is invertible. Indeed, write

$$B = A[I - A^{-1}(A-B)].$$

Since $\|A^{-1}(A-B)\| < 1$, using Theorem 2.2, it follows that $I - A^{-1}(A-B)$ is invertible. Now since A is invertible it follows that $A[I - A^{-1}(A-B)]$ is invertible, too.

Definition 2.4. If $A : \mathscr{B} \to \mathscr{B}$ is a bounded linear operator, $N(A)$, $R(A)$, $\sigma(A)$, and $\rho(A)$ stand for the kernel, range, spectrum, and the resolvent of A, respectively, defined by

$$N(A) = \left\{u \in \mathscr{B} : \ Au = 0\right\},$$

$$R(A) = \left\{Au : \ u \in \mathscr{B}\right\},$$

$$\sigma(A) = \left\{\lambda \in \mathbb{C} : \ \lambda I - A \ \text{ is not invertible}\right\},$$

and $\rho(A)$ is the collection of all $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is one-to-one ($N(A - \lambda I) = 0$), onto ($R(A - \lambda I) = \mathscr{B}$), and bounded.

Remark 2.2. Note that $\lambda \in \sigma(A)$ if and only if at least one of the following assertions holds true:

(i) $R(\lambda I - A) \neq \mathscr{B}$;

(ii) $\lambda I - A$ is not one-to-one.

Note that if (ii) of Remark 2.2 holds, λ is called an eigenvalue of the operator A with corresponding eigenspace $N(A - \lambda I)$. Therefore, if $0 \neq u \in N(\lambda I - A)$ is an eigenvalue then $Au = \lambda u$.

Example 2.5. Let $q : [\alpha,\beta] \to \mathbb{C}$ be a continuous function. Define the bounded linear operator M_q on $\mathscr{B} = L^2([\alpha,\beta])$ by

$$(M_q\phi)(s) = q(s)\phi(s), \ \ \forall s \in [\alpha,\beta].$$

It can be shown that $\lambda I - M_q$ is invertible on $L^2([\alpha,\beta])$ if and only if

$$\lambda - q(s) \neq 0, \ \ \forall s \in [\alpha,\beta]. \tag{2.8}$$

The inverse $(\lambda I - M_q)^{-1}$ of $\lambda I - M_q$ is defined by

$$\left[(\lambda I - M_q)^{-1}\right]\psi(s) = \left[\frac{1}{\lambda - q(s)}\right]\psi(s)$$

with the following estimate:

$$\left\|(\lambda I - M_q)^{-1}\right\| \leq \max_{s\in[\alpha,\beta]}\left[\frac{1}{|\lambda - q(s)|}\right].$$

The spectrum $\sigma(M_q)$ of M_q is given by

$$\sigma(M_q) = \{q(s) : s \in [\alpha,\beta]\}.$$

2.2.1.3 Compact Operators

Definition 2.5. A bounded linear operator A on $\mathscr{B}$ is said to be compact if it maps the unit ball U $(U = \{x \in \mathscr{B} : \|x\| \leq 1\})$ into a set whose closure is compact.

Equivalently,

Definition 2.6. A bounded linear operator A on $\mathscr{B}$ is said to be compact if for each sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{B}$ with $\|x_n\| \leq 1$ for each $n \in \mathbb{N}$, the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a subsequence which converges in $\mathscr{B}$.

The collection of all compact operators on $\mathscr{B}$ is denoted $\mathscr{K}(\mathscr{B})$. The next theorem shows that $\mathscr{K}(\mathscr{B})$ is a two-sided ideal of $B(\mathscr{B})$. Moreover, $\mathscr{K}(\mathscr{B})$ is closed for the operator norm.

Theorem 2.3. *If $A,B \in B(\mathscr{B})$ are compact linear operators, then*
(i) αA is compact;
(ii) $A+B$ is compact;
(iii) if $C \in B(\mathscr{B})$, then AC and CA are compact.

Proof. (i) is straightforward.

(ii) Let $(u_n) \in \mathscr{B}$ with $\|x_n\| \leq 1$. Since A is compact, $(Au_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(Au_{n_k})_{k\in\mathbb{N}}$.

Similarly, $(Bu_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(Bu_{n_k})_{k\in\mathbb{N}}$. Therefore, $((A+B)u_{n_k})_{k\in\mathbb{N}}$ converges.

(iii) Let $(v_n)_{n\in\mathbb{N}} \subset \mathscr{B}$ with $\|v_n\| \leq 1$ for each $n \in \mathbb{N}$. Thus $(Cv_n)_{n\in\mathbb{N}}$ is bounded. Now since A is compact, it is clear $(ACv_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Let $(w_n)_{n\in\mathbb{N}} \subset \mathbb{H}$ with $\|w_n\| \leq 1$ for each $n \in \mathbb{N}$. Now since A is compact, $(Aw_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say $(Aw_{n_k})_{k\in\mathbb{N}}$. Now by the continuity of C it follows that $(CAw_{n_k})_{k\in\mathbb{N}}$ converges.

Example 2.6. In $\mathscr{B} = L^2[a,b]$, define the integral operator A by

$$(Af)(t) := \int_a^b V(t,\tau)f(\tau)d\tau \text{ for each } f \in L^2[a,b].$$

Assuming that $V \in L^2([a,b]\times[a,b])$, it can be shown that A is compact.

Remark 2.3. (i) A bounded linear operator is of finite–rank if its image is a finite–dimensional Banach space.

(ii) A finite-rank operator is compact since all balls are pre-compact in a finite–dimensional Banach space.

(iii) Compact operators Hilbert spaces are uniform operator norm limits of finite–rank operators, and conversely.

(iv) If a sequence of compact operators converges to some bounded operator for the operator norm, then the limit is also a compact operator.

(v) If A is a compact operator, so is its adjoint A^*.

2.2.1.4 Hilbert–Schmidt Operators

An important subclass of compact operators consists of the so-called Hilbert–Schmidt operators. Here we study basic properties of Hilbert–Schmidt operators.

Definition 2.7. Let $(e_n)_{n=1,2...}$ be an orthonormal basis for the Hilbert space $\mathscr{H}$. An operator $A \in B(\mathscr{H})$ is called a Hilbert–Schmidt if

$$\|A\|_2 := \left(\sum_{n=1}^{\infty} \|Ae_n\|^2\right)^{1/2} < \infty. \tag{2.9}$$

If Eq. (2.9) holds, the number $\|A\|_2$ is called the Hilbert–Schmidt norm of A. We denote the class of Hilbert–Schmidt linear operators on $\mathscr{H}$ by $\mathbb{L}_2(\mathscr{H})$. More generally, if $\mathscr{H}'$ is another Hilbert space, we denote the collection of all Hilbert–Schmidt operators from $\mathscr{H}$ to $\mathscr{H}'$ by $\mathbb{L}_2(\mathscr{H},\mathscr{H}')$.

Example 2.7. Let $(e_n)_{n\geq 1}$ be the canonical orthonormal basis for the Hilbert space $\mathscr{H} = l^2$ and let A be the operator defined by

$$Ax = \sum_{n=1}^{\infty} \frac{1}{n} \langle x, e_n \rangle\, e_n.$$

Clearly, $\|A\|_2 = \dfrac{\pi}{\sqrt{6}}$, and hence A is a Hilbert–Schmidt operator.

More generally:

Example 2.8. Let $\mathscr{H}$ be a Hilbert space and let A be the diagonal operator defined by

$$Au = \sum_{n=1}^{\infty} \alpha_n \langle u, e_n \rangle \, e_n, \;\; \forall u \in \mathbb{H},$$

where $(e_n)_{n\geq 1}$ is an orthonormal basis for $\mathscr{H}$.

Clearly, $Ae_n = \alpha_n e_n$, $\forall n = 1, 2, \ldots$, and $\|A\|_2 = \left(\sum_{n=1}^{\infty} |\alpha_n|^2 \right)^{1/2}$. Hence, A is Hilbert–Schmidt if and only if

$$\|A\|_2 = \left(\sum_{n=1}^{\infty} |\alpha_n|^2 \right)^{1/2} < \infty.$$

For instance, the operator B defined on $\mathbb{H}$ by

$$Bu = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \langle u, e_n \rangle \, e_n, \;\; \forall u \in \mathscr{H}$$

is not Hilbert–Schmidt while the operator C,

$$Cu = \sum_{n=1}^{\infty} \frac{1}{n^2} \langle u, e_n \rangle \, e_n, \;\; \forall u \in \mathscr{H}$$

is since $\|C\|_2 = \dfrac{\pi^2}{3\sqrt{10}}$.

Remark 2.4. Let us notice that the Hilbert–Schmidt norm $\|\cdot\|_2$ is independent of the orthonormal basis $(e_n)_{n\geq 1}$ considered in the Definition 2.7.

Proposition 2.4. *A bounded linear operator A on $\mathscr{H}$ is Hilbert–Schmidt if and only if its adjoint A^* is. Furthermore, $\|A\| \leq \|A\|_2$, and $\|A\|_2 = \|A^*\|_2$.*

Proof. Since A is a Hilbert–Schmidt operator then there exists an orthonormal basis $(e_n)_{n\geq 1}$ for $\mathscr{H}$ such that $\sum_{n=1} \|Ae_n\|^2 < \infty$.

If $(f_n)_{n\geq 1}$ is another orthonormal basis for $\mathscr{H}$, then

$$\begin{aligned}
\sum_{n\geq 1} \|A^* f_n\|^2 &= \sum_{n\geq 1} \sum_{m\geq 1} |\langle A^* f_n, e_m \rangle|^2 \\
&= \sum_{m\geq 1} \sum_{n\geq 1} |\langle f_n, Ae_m \rangle|^2 \\
&= \sum_{n\geq 1} \|Ae_n\|^2
\end{aligned}$$

and hence A^* is a Hilbert–Schmidt operator with $\|A^*\|_2 = \|A\|_2$.

The converse can be proved using similar arguments as above.

Now

$$
\begin{aligned}
\|Au\|^2 &= \sum_{m\geq 1} |\langle f_m, Au\rangle|^2 \\
&\leq \|u\|^2 \sum_{m\geq 1} \|A^* f_m\|^2
\end{aligned}
$$

and hence $\|A\| \leq \|A^*\|_2 = \|A\|_2$.

Proposition 2.5. *Let $A, B \in B(\mathscr{H})$. Suppose that $A \in \mathbb{L}_2(\mathscr{H})$, then both AB and BA are in $\mathbb{L}_2(\mathscr{H})$.*

Proof. Since A is a Hilbert–Schmidt operator then there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for $\mathscr{H}$ such that $\sum\limits_{n=1} \|Ae_n\|^2 < \infty$. We have

$$
\sum_{n=1}^{\infty} \|BAe_n\|^2 \leq \|B\|^2 \sum_{n=1} \|Ae_n\|^2 < \infty,
$$

hence $BA \in \mathbb{L}_2(\mathscr{H})$.

To complete the proof it remains to show that AB is a Hilbert–Schmidt operator. Indeed, $AB = (B^*A^*)^* \in \mathbb{L}_2(\mathscr{H})$.

Theorem 2.4. *Every Hilbert–Schmidt operator is compact and is the limit in $\|\cdot\|_2$–norm of a sequence of operators of finite–rank.*

Proof. We refer the reader to [176].

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset and let $A \in B(L^2(\Omega))$, then $A \in \mathbb{L}_2(L^2(\Omega))$ if and only if there exists a kernel $V \in L^2(\Omega \times \Omega)$ such that*

$$
A\phi(x) = \int_{\Omega} V(x,y)\phi(y)dy \tag{2.10}
$$

for all $x \in \Omega$ and $\phi \in L^2(\Omega)$.

The adjoint A^* of A is the integral operator with kernel W defined by $W(x,y) := \overline{V(y,x)}$. Thus,

$$
A^*\psi(x) = \int_{\Omega} W(x,y)\psi(y)dy \tag{2.11}
$$

for all $x \in \Omega$ and $\psi \in L^2(\Omega)$.

Proof. See for instance [176].

Remark 2.5. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for a separable Hilbert space $\mathscr{H}$ and let A and B be Hilbert–Schmidt operators on $\mathscr{H}$, respectively. Define

$$\langle\langle A,B\rangle\rangle = \sum_{n=1}^{\infty} \langle Ae_n, Be_n\rangle. \tag{2.12}$$

It can be easily shown that $\langle\langle A,B\rangle\rangle$ is an inner product on $\mathbb{L}_2(\mathscr{H})$ and that $|A| = (\langle\langle A,A\rangle\rangle)^{\frac{1}{2}}$. In particular, $(\mathbb{L}_2(\mathscr{H}), \langle\langle\cdot,\cdot\rangle\rangle)$ is a Hilbert space.

2.2.2 Unbounded Linear Operators

Definition 2.8. An unbounded linear operator A from $\mathscr{B}$ in $\mathscr{B}'$ is a pair $(D(A),A)$ consisting of a subspace $D(A) \subset \mathscr{B}$ (called the domain of A) and a (possibly not continuous) linear transformation $A : D(A) \subset \mathscr{B} \to \mathscr{B}'$.

2.2.3 Examples of Unbounded Operators

Example 2.9. [51, 130] Set $\mathscr{B} = \mathscr{B}' = L^2(\mathbb{R})$ and consider the one–dimensional Laplace operator defined by

$$D(A) = W^{2,2}(\mathbb{R}) = H^2(\mathbb{R}) \;\text{ and }\; Au = -u''$$

for all $u \in H^2(\mathbb{R})$.

Recall that $L^2(\mathbb{R})$ is endowed with the norm defined by

$$\|\psi\|_2^2 := \int_{-\infty}^{+\infty} |\psi(t)|^2\,dt, \;\; \forall\psi \in L^2(\mathbb{R}).$$

Now, consider the sequence of functions defined by $\psi_n(t) = e^{-n|t|}$, $n = 1,2,...$. Clearly, for each $n = 1,2,...$, $\psi_n \in D(A) = H^2(\mathbb{R})$. Furthermore,

$$\|\psi_n\|_2^2 = \int_{-\infty}^{+\infty} e^{-2n|t|}dt = \frac{1}{n}$$

and

$$\|A\psi_n\|_2^2 = \int_{-\infty}^{+\infty} n^4 e^{-2n|t|}dt = n^3.$$

Therefore, $\dfrac{\|A\psi_n\|_2}{\|\psi_n\|_2} = n \to \infty$ as n goes to ∞, that is, A is an unbounded linear operator on $L^2(\mathbb{R})$.

Example 2.10. [51, 130] Set $\mathscr{B} = \mathscr{B}' = L^2(0,1)$ and consider the derivative operator defined by

$$D(A) = C^1(0,1) \;\text{ and }\; Au = u'$$

for all $u \in C^1(0,1)$, where $C^1(0,1)$ is the collection of continuously differentiable functions over $(0,1)$.

Consider the sequence of functions defined by $\phi_n(t) = t^n,\ \ n = 1,2,....$ Clearly, for each $n = 1,2,...,$ $\phi_n \in C^1(0,1)$. Furthermore,

$$\left\|\phi_n\right\|_2^2 = \int_0^1 t^{2n}dt = \frac{1}{2n+1},$$

and

$$\|A\phi_n\|_2^2 = \int_0^1 n^2 t^{2n-2}\,dt = \frac{n^2}{2n-1}.$$

Here again,

$$\frac{\|A\phi_n\|_2}{\|\phi_n\|_2} = n\sqrt{\frac{2n+1}{2n-1}} \to \infty$$

as n goes to ∞, that is, A is an unbounded linear operator on $L^2(\mathbb{R})$.

Example 2.11. (Multiplication Operator) Let $\mathscr{O} \subset \mathbb{R}$ be an arbitrary interval and let $C_0(\mathscr{O})$ denote the collection of all continuous functions $u : \mathscr{O} \to \mathbb{C}$ satisfying $\forall \varepsilon > 0$ then there exists a compact interval $I_\varepsilon \subset \mathscr{O}$ such that

$$|u(s)| < \varepsilon,\ \ \forall s \in \mathscr{O}\backslash I_\varepsilon.$$

Then define the multiplication operator M_γ on $C_0(\mathscr{O})$ by

$$\begin{cases} D(M_\gamma) = \{u \in C_0(\mathscr{O}) :\ \gamma u \in C_0(\mathscr{O})\}, \\ \\ \qquad M_\gamma u = \gamma(x)u, \quad \forall u \in D(M_\gamma), \end{cases}$$

where $\gamma : \mathscr{O} \to \mathbb{C}$ is continuous.

In view of the above, M_γ is an unbounded linear operator on $C_0(\mathscr{O})$. Moreover, one can show that M_γ is bounded if and only if γ is bounded. In that event,

$$\left\|M_\gamma\right\| = \|\gamma\|_\infty := \sup_{s\in\mathscr{O}} |\gamma(s)|.$$

Definition 2.9. If $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ is an unbounded linear operator on $\mathscr{B}$, then its graph is defined by

$$\mathscr{G}(A) = \Big\{(x, Ax) \in \mathscr{B} \times \mathscr{B} :\ x \in D(A)\Big\}.$$

Definition 2.10. If A, B are unbounded linear operators on $\mathscr{B}$, then A is said to be an extension of B if $D(B) \subset D(A)$ and $Au = Bu$ for all $u \in D(B)$. In that event, we denote it by $B \subset A$. Moreover, $B \subset A$ if and only if $\mathscr{G}(B) \subset \mathscr{G}(A)$.

The notion of graph of an operator is very important as it enables us to deal with the closure of an operator.

2.2.3.1 Closed and Closable Linear Operators

Definition 2.11. A linear operator $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ is called closed if its graph $\mathscr{G}(A) \subset \mathscr{B} \times \mathscr{B}$ is closed.

The closedness of an unbounded linear operator A can be characterized as follows: if $u_n \in D(A)$ such that $u_n \to u$ and $Au_n \to v$ in $\mathscr{B}$ as $n \to \infty$, then $u \in D(A)$ and $Ax = v$.

Example 2.12. Every bounded linear operator $A : \mathscr{B} \to \mathscr{B}$ is closed.

Proof. Suppose $(u_n)_{n\in\mathbb{N}} \in D(A)$ such that $u_n \to u$ with $Au_n \to v$ in $\mathscr{B}$ as $n \to \infty$. Now since A is bounded, therefore $D(A) = \mathscr{B}$. Again, from the continuity of A it is clear that $u \in \mathscr{B}$ and $Au = v$.

Example 2.13. Let $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ be a closed linear operator and let $B \in B(\mathscr{B})$, then $A + B$ is closed.

Proof. Suppose $(u_n)_{n\in\mathbb{N}} \in D(A+B) = D(A)$ such that $u_n \to u$ and $(A+B)u_n \to v$ in $\mathscr{B}$ as $n \to \infty$.

Now since B is bounded it follows that $Au_n \to v - Bx$ in $\mathscr{B}$ as $n \to \infty$. Since A is closed, then $u \in D(A)$ and $Au = v - Bu$.

Example 2.13 can be illustrated as follows: Let $\mathscr{B} = L^2(\mathbb{R}^n)$ and define A and B by

$$D(A) = W^{2,2}(\mathbb{R}^n) = H^2(\mathbb{R}^n) \text{ and } Au = -\Delta u, \ \forall u \in H^2(\mathbb{R}^n),$$

and

$$D(B) = \left\{ u \in L^2(\mathbb{R}^n) : \gamma(x)u \in L^2(\mathbb{R}^n) \right\} \text{ and } Bu = \gamma(x)u, \ \forall u \in D(B),$$

where Δ is the n-dimensional Laplace operator defined by

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

and that $\gamma \in L^\infty(\mathbb{R}^n)$.

It is then clear that $B = M_\gamma$ is a bounded linear operator and therefore $-\Delta + \gamma$ is closed. Furthermore, $D(-\Delta + \gamma) = H^2(\mathbb{R}^n)$.

Example 2.14. The multiplication operator M_γ given in Example 2.11 is closed.

Definition 2.12. An operator $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ is said to be closable if it has a closed extension.

It is well-known that A is closable if $\overline{\mathscr{G}(A)}$ is a graph. Equivalently, $u_n \in D(A)$, $u_n \to 0$ and $Au_n \to v$ then $v = 0$.

If A is closable, then its smallest closed extension is called the closure of A and is denoted by $\overline{A}$. The operator $\overline{A}$ is defined by

$$\begin{cases} D(\overline{A}) = \{u \in \mathscr{B} : \exists u_n \in D(A),\ u_n \to u,\ Au_n \ \text{converges}\}, \\ \overline{A}u = \lim Au_n,\ \forall u \in D(\overline{A}). \end{cases}$$

Moreover, the closure $\overline{A}$ of A satisfies $\mathscr{G}(\overline{A}) = \overline{\mathscr{G}(A)}$.

Example 2.15. Suppose $\mathscr{B} = L^2(\mathbb{R}^n)$ and consider the linear operator A defined by

$$D(A) = C_0^\infty(\mathbb{R}^n) \ \text{and} \ Au = -\Delta u, \ \text{for all} \ u \in C_0^\infty(\mathbb{R}^n).$$

Clearly, A is closable and its closure is given by

$$D(\overline{A}) = H^2(\mathbb{R}^n) \ \text{and} \ \overline{A}u = -\Delta u, \ \text{for all} \ u \in H^2(\mathbb{R}^n).$$

2.2.3.2 Spectral Theory for Unbounded Linear Operators

If $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ is a closed linear operator on $\mathscr{B}$, then $\rho(A)$ the resolvent set of A is defined by

$$\rho(A) = \left\{\lambda \in \mathbb{C} : \lambda I - A \text{ is one-to-one, and } (\lambda I - A)^{-1} \in B(\mathscr{B})\right\},$$

and $\sigma(A)$ the spectrum of A is the complement of the resolvent set $\rho(A)$ in $\mathbb{C}$.

Now, if $\lambda \in \rho(A)$, then the operator–valued function $R(\lambda, A) := (\lambda I - A)^{-1} : \rho(A) \mapsto B(\mathscr{B})$ is called the resolvent of the operator A. It should be mentioned that $\rho(A) \neq \emptyset$ if A is closed.

As for bounded linear operators, the spectrum of an unbounded operator can be divided into three disjoint subsets of the complex plane, that is,

$$\sigma(A) = \sigma_c(A) \cup \sigma_r(A) \cup \sigma_p,$$

where $\sigma_c(A), \sigma_r(A), \sigma_p(A)$ are respectively the continuous spectrum, the residual spectrum, and the point spectrum of the operator A defined by:

(i) $\lambda \in \sigma_c(A)$ if $\lambda \in \mathbb{C}$, $\lambda I - A$ is one-to-one, and $\overline{R(\lambda I - A)} = \mathscr{B}$;
(ii) $\lambda \in \sigma_r(A)$ if $\lambda \in \mathbb{C}$, $\lambda I - A$ is one-to-one, and $\overline{R(\lambda I - A)} \neq \mathscr{B}$; and
(iii) $\lambda \in \sigma_p(A)$ if $\lambda \in \mathbb{C}$, and $\lambda I - A$ is not one-to-one.

Example 2.16. Fix $\Theta, L > 0$. In $\mathscr{B} := L^2(0, L)$ equipped with its natural topology $\|\cdot\|_2$, define the operator A by

$$A\varphi := -\Theta u'', \ \forall \varphi \in D(A),$$

where $D(A) := H_0^1(0, L) \cap H^2(0, L)$.

The resolvent and spectrum of the linear operator A are respectively given by

$$\rho(A) = \mathbb{C} - \left\{\frac{\Theta \pi^2}{L^2} n^2 : \ n = 1, 2, 3, ...\right\}$$

and

$$\sigma(A) = \sigma_p(A) = \left\{ \frac{\Theta\pi^2}{L^2} n^2 : \ \ n = 1,2,3,... \right\}.$$

Proposition 2.7. *If $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ is a closed linear operator and if $\lambda, \mu \in \rho(A)$, then for any $\lambda \in \rho(A)$, the operator $R(\lambda, \mathscr{B})$ is a bounded linear operator $\mathscr{B}$.*

Proof. Let $\lambda \in \rho(A)$. Clearly, $R(\lambda I - A) = D((\lambda I - A)^{-1})$ is dense in $\mathscr{B}$ and there exists $K > 0$ such that

$$\|(\lambda I - A)u\| \geq K\|u\| \ \text{ for all } \ u \in D(A).$$

To complete the proof, we have to show that $R(\lambda I - A) = \mathscr{B}$. Indeed, let $(u_n)_{n\in\mathbb{N}} \subset D(A)$ and suppose that $(\lambda I - A)u_n \to v$ as $n \to \infty$. Using the above-mentioned inequality it follows that there exists some $u \in \mathscr{B}$ such that $u_n \to u$ as $n \to \infty$. Since A is closed it follows that $u \in D(A)$ and $(\lambda I - A)u = v$. Consequently, by the density assumption $\overline{R(\lambda I - A)} = \mathscr{B}$, we must have $R(\lambda I - A) = \mathscr{B}$.

Proposition 2.8. *Let A and B be two (possibly unbounded) closed linear operators on $\mathscr{B}$.*

(i) If $\lambda, \mu \in \rho(A)$, then

$$R(\lambda,A) - R(\mu,A) = (\mu - \lambda)R(\lambda,A)\, R(\mu,A). \tag{2.13}$$

Furthermore, $R(\lambda,A)$ and $R(\mu,A)$ commute.

(ii) If $D(A) \subset D(B)$, then for all $\lambda \in \rho(A) \cap \rho(B)$ we have

$$R(\lambda,A) - R(\lambda,B) = R(\lambda,A)(A - B)R(\lambda,B). \tag{2.14}$$

(iii) If $D(A) = D(B)$, then for all $\lambda \in \rho(A) \cap \rho(B)$ we have

$$R(\lambda,A) - R(\lambda,B) = R(\lambda,A)(A - B)R(\lambda,B) = R(\lambda,B)(A - B)R(\lambda,A). \tag{2.15}$$

Proof. (i) Write

$$\begin{aligned} R(\lambda,A) - R(\mu,A) &= R(\lambda,A)[(\mu I - A) - (\lambda I - A)]R(\mu,A) \\ &= (\mu - \lambda)R(\lambda,A)\, R(\mu,A). \end{aligned}$$

Now, the second statement is obtained from the first one by

$$\begin{aligned} R(\lambda,A)\, R(\mu,A) &= \frac{1}{\mu - \lambda}[R(\lambda,A) - R(\mu,A)] \\ &= \frac{1}{\lambda - \mu}[R(\mu,A) - R(\lambda,A)] \\ &= R(\mu,A)\, R(\lambda,A). \end{aligned}$$

(ii) Write

$$\begin{aligned} R(\lambda,A) - R(\lambda,B) &= R(\lambda,A)[(\lambda I - B) - (\lambda I - A)]R(\lambda,B) \\ &= R(\lambda,A)(A-B)R(\lambda,B). \end{aligned}$$

(iii) Write

$$\begin{aligned} R(\lambda,A) - R(\lambda,B) &= R(\lambda,A)[(\lambda I - B) - (\lambda I - A)]R(\lambda,B) \\ &= R(\lambda,A)(A-B)R(\lambda,B) \\ &= R(\lambda,B)(A-B)R(\lambda,A). \end{aligned}$$

Theorem 2.5. *If $A : D(A) \subset \mathscr{H} \to \mathscr{B}$ is a closed linear operator, then $\rho(A)$ is an open subset of $\mathbb{F}$. Therefore, $\sigma(A)$ is closed. Namely, if $\lambda \in \rho(A)$, then $\mu \in \rho(A)$ for all $\mu \in \mathbb{F}$ such that $|\lambda - \mu| < \|R(\lambda,A)\|^{-1}$ and for those μ, the following holds:*

$$R(\mu,A) = \sum_{n\in\mathbb{N}} (\lambda-\mu)^n R(\lambda,A)^{n+1}.$$

If $A \in B(\mathscr{H})$, then $\{\mu \in \mathbb{F} : |\mu| > \|A\|\} \subset \rho(A)$. Moreover, the spectrum of A is compact, and

$$R(\mu,A) = \sum_{n\in\mathbb{N}} \lambda^{-n-1}A^n \ \textit{for all} \ |\mu| > \|A\|.$$

Proof. See [186] for details.

2.2.3.3 Symmetric and Self-Adjoint Linear Operators

Definition 2.13. If $A : D(A) \subset \mathscr{H} \to \mathscr{H}$ is a densely defined linear operator, then its adjoint denoted A^* is defined in a unique fashion by

$$D(A^*) = \Big\{ v \in \mathscr{H} : u \mapsto \langle Au, v\rangle \ \text{is} \ \mathscr{H}-\text{continuous over}\, D(A) \Big\},$$

and

$$\langle Au, v\rangle = \langle u, A^* v\rangle, \ \forall u \in D(A), \ v \in D(A^*).$$

Define the mappings $U : \mathscr{H} \times \mathscr{H} \mapsto \mathscr{H} \times \mathscr{H}$ and $V : \mathscr{H} \times \mathscr{H} \mapsto \mathscr{H} \times \mathscr{H}$ by setting $U(x,y) = (y,-x)$ and $V(x,y) = (y,x)$ for all $(x,y) \in \mathscr{H} \oplus \mathscr{H}$. Clearly, U and V are isomorphisms from $\mathscr{H} \oplus \mathscr{H}$ onto $\mathscr{H} \oplus \mathscr{H}$. Furthermore, their inverses are defined by

$$U^{-1}(x,y) = (-y,x)$$

and

$$V^{-1}(x,y) = (y,x)$$

for all $x, y \in \mathscr{H}$.

If A is a densely defined (possibly unbounded) linear operator, that is, $\overline{D(A)} = \mathscr{H}$, one can easily see that

$$\mathscr{G}(A^*) = U\Big(\mathscr{G}(A)^{\perp}\Big) = \Big(U\mathscr{G}(A)\Big)^{\perp}. \tag{2.16}$$

Proposition 2.9. *[176] If $A : D(A) \subset \mathscr{H} \to \mathscr{H}$ is a densely defined ($\overline{D(A)} = \mathscr{H}$) unbounded linear operator, then*

(i) A^* is closed;

(ii) A is closable if and only if A^* is densely defined; in this case $\overline{A} = (A^*)^*$; and

(iii) if A is closable, then $(\overline{A})^* = A^*$.

Proof. (i) Using the identity Eq. (2.16) it follows that $\mathscr{G}(A^*) = \Big(U\mathscr{G}(A)\Big)^{\perp}$. Consequently, $\mathscr{G}(A^*)$ is closed.

(ii) Using the fact

$$\begin{aligned}
\overline{\mathscr{G}(A)} &= \mathscr{G}(A)^{\perp\perp} \\
&= \Big(U^{-1}\mathscr{G}(A^*)\Big)^{\perp} \\
&= \{(u,v) \in \mathscr{H} \oplus \mathscr{H} : \ \langle u, A^*z\rangle - \langle v, z\rangle = 0 \ \text{ for all } \ z \in D(A^*)\}
\end{aligned}$$

it follows that $(0,v) \in \overline{\mathscr{G}(A)}$ if and only if $v \in D(A^*)^{\perp}$. And hence $(0,v) \in \overline{\mathscr{G}(A)}$ yields $v = 0$ if and only if $\overline{D(A^*)} = \mathscr{H}$. Therefore, $\overline{\mathscr{G}(A)}$ is a graph if and only if the linear operator A^* is densely.

Now, if $D(A^*)$ is dense, then

$$\mathscr{G}(A^{**}) = \Big(U^{-1}\mathscr{G}(A^*)^{\perp}\Big) = \Big(U^{-1}U\mathscr{G}(A)^{\perp\perp}\Big) = \overline{\mathscr{G}(A)} = \mathscr{G}(\overline{A}).$$

(iii) Suppose A is closable. Now,

$$\mathscr{G}(A^*) = U\Big(\mathscr{G}(A)^{\perp}\Big) = U\Big(\overline{\mathscr{G}(A)^{\perp}}\Big) = U\Big(\mathscr{G}(\overline{A})\Big)^{\perp} = \mathscr{G}(A^*),$$

and hence $A^* = (\overline{A})^*$.

Proposition 2.10. *[176] If A, B are densely defined unbounded linear operators on $\mathscr{H}$, then*

(i) $A^*B^* \subset (BA)^*$;

(ii) if $B \in B(\mathscr{H})$, then $A^*B^* = (BA)^*$; and

(iii) if $A + B$ is densely defined, we have $(A+B)^* \supset A^* + B^*$.

Proof. (i) Let us show that the operators A^*B^* and BA are adjoint to each other. Indeed, let $u \in D(A^*B^*)$ and $v \in D(BA)$. Clearly, $u \in D(B^*)$ such that $B^*u \in D(A^*)$. Similarly, $v \in D(A)$ such that $Av \in D(B)$. Using the definition of the adjoint it follows

$$\langle A^*B^*u, v\rangle = \langle B^*u, Av\rangle = \langle u, BAv\rangle.$$

(ii) Using (i) it is enough to show that $D(BA)^*) \subset D(A^*B^*)$. Indeed, let $u \in D(BA)^*)$. Now since B^* is bounded it follows that for all $v \in D(BA) = D(A)$, we obtain

$$\langle (BA)^*u, v\rangle = \langle u, BAv\rangle = \langle B^*u, v\rangle,$$

and hence $B^*u \in D(A^*)$, that is, $u \in D(A^*B^*)$.

(iii) Let $u \in D(A^* + B^*) = D(A^*) \cap D(B^*)$. Clearly, for all $v \in D(A+B) = D(A) \cap D(B)$, we have

$$\langle (A^* + B^*)u, v\rangle = \langle A^*u, v\rangle + \langle B^*u, v\rangle = \langle u, Av\rangle + \langle u, Bv\rangle = \langle u, (A+B)v\rangle$$

and hence $u \in D((A+B)^*)$ and $(A+B)^*u = A^*u + B^*u$.

Definition 2.14. If $A : D(A) \subset \mathscr{H} \to \mathscr{H}$ is a densely defined operator on $\mathscr{H}$, then

(i) A is symmetric if $A \subset A^*$.

(ii) A is self-adjoint if $A = A^*$.

Example 2.17. Let $\mathscr{H} = L^2[0,1]$ and define the linear operator A by

$$D(A) = \{u \in L^2[0,1] : u \in C^1[0,1],\ \ u(0) = u(1) = 0\}$$

and

$$Au = iu' \ \text{ for all } \ u \in D(A).$$

It is not hard to see that

$$A^*u = iu' \ \text{ for all } \ u \in D(A^*)$$

where

$$D(A^*) = \{u : \ \ u \ \text{ is absolutely continuous, } \ u' \in L^2[0,1]\}.$$

Therefore, $A \subset A^*$, that is, A is symmetric. It should also be noted that A is not closed. It is obviously closable and has a closure $\overline{A}$ defined by

$$\overline{A}u = iu' \ \text{ for all } \ u \in D(\overline{A})$$

where

$$D(\overline{A}) = \{u : \ \ u \ \text{ is absolutely continuous, } \ u' \in L^2[0,1], \ \ u(0) = u(1) = 0\}.$$

Example 2.18. Let $\mathscr{H} = L^2[0,1]$ and define the linear operator B by

$$D(B) = \{u : \ \ u \ \text{ is absolutely continuous, } \ u' \in L^2[0,1], \ \ u(0) = u(1) = 0\}$$

and

$$Au = iu' \ \text{ for all } \ u \in D(B).$$

It is not hard to see that $B \subset B^*$, that is, B is symmetric. Moreover, it can be shown that $B = B^*$.

Proposition 2.11. *Every symmetric operator A on $\mathscr{H}$ is closable.*

Proof. Clearly A is closable since $A \subset A^*$ and A^* is closed by Proposition 2.9 (i). Now for all $u, v \in D(A)$ one can find sequences $u_n, v_n \subset D(A)$ such that $u_n \to u$ and $v_n \to v$ and $Au_n \to \overline{A}u$, $Av_n \to \overline{A}v$ as $n \to \infty$. Since A is symmetric it follows that

$$\langle \overline{A}u, v \rangle = \lim_{n\to\infty} \langle Au_n, v_n \rangle = \lim_{n\to\infty} \langle u_n, Av_n \rangle = \langle u, \overline{A}v \rangle.$$

Since $D(\overline{A})$ is dense it follows that $\overline{A}$ is symmetric, too.

Remark 2.6. Notice that a symmetric operator A is called essentially self-adjoint if it has a unique self-adjoint extension.

Theorem 2.6. *Let $A : D(A) \subset \mathscr{H} \mapsto \mathscr{H}$ be a self-adjoint operator, then*

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A).$$

Proof. We refer the reader to [69].

Theorem 2.7. *[69] Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be a spectral family of orthoprojections E_λ, that is, $E_\lambda \le E_\mu$ for $\lambda \le \mu$ and $E_\lambda \to 0$ as $\lambda \to -\infty$, $E_\lambda \to I$ as $\lambda \to \infty$ (in the strong sense) and $E_{\lambda+0} = E_\lambda$. Now let A be the operator defined by*

$$D(A) = \{u \in \mathscr{H} : \int_{-\infty}^{\infty} \lambda^2 d\langle E_\lambda x, x \rangle < \infty\}$$

and

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda$$

that is for each $u \in D(A)$, we have

$$Au = \int_{-\infty}^{\infty} \lambda dE_\lambda u.$$

Then A is a self-adjoint linear operator on $\mathscr{H}$ and

$$\|Au\|^2 = \int \lambda^2 d\langle E_\lambda u, u \rangle.$$

2.3 Sectorial Linear Operators

An important class of (unbounded) linear operators is that of sectorial linear operators. Such a class of operators will play an important role throughout this book.

2.3.1 Basic Definitions

Definition 2.15. A linear operator $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ (not necessarily densely defined) is said to be sectorial if the following hold: There exist constants $\zeta \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, and $M > 0$ such that

(i) $\rho(A) \supset S_{\theta,\zeta} := \{\lambda \in \mathbb{C} : \lambda \neq \zeta,\ \ |\arg(\lambda-\zeta)| < \theta\}$, and
(ii)$\|R(\lambda,A)\| \leq \dfrac{M}{|\lambda-\zeta|}$ for each $\lambda \in S_{\theta,\zeta}$.

Let us notice that since the resolvent of a sectorial operator A is nonempty, then A is closed. Therefore, the space $(D(A), \|\cdot\|_A)$ where

$$\|x\|_A = \|x\| + \|Ax\|$$

for each $x \in D(A)$, is a Banach space.

Note that the norm $\|\cdot\|_A$ which depends heavily on the operator A is called the graph norm of A.

Proposition 2.12. *[129] Let A be a linear operator on $\mathscr{B}$ such that $\rho(A)$ contains the half-plane $\{\lambda \in \mathbb{C} : \Re e\lambda \geq \zeta\}$, and*

$$\|\lambda R(\lambda,A)\| \leq M, \quad \Re e\lambda \geq \zeta,$$

with $\zeta \in \mathbb{R}$ and $M > 0$. Then A is sectorial.

2.3.2 Examples of Sectorial Operators

Example 2.19. In $\mathscr{B} = L^p(0,1)$ $(p \geq 1)$ equipped with its natural norm, define the linear operator A by

$$Au = u'' \ \text{ for each } \ u \in D(A) = \left\{u \in W^{2,p}(0,1) : u(0) = u(1) = 0\right\}.$$

Then the linear operator A defined above is sectorial.

Example 2.20. In $\mathscr{B} = C[0,1]$ equipped with the sup norm, define the linear operator A by

$$Au = u'' \ \text{ for each } \ u \in D(A) = \left\{u \in C^2[0,1] : u(0) = u(1) = 0\right\}.$$

Then the linear operator A defined above is sectorial.

Example 2.21. Let $\mathscr{O} \subset \mathbb{R}^n$ be a bounded open subset with C^2 boundary $\partial\mathscr{O}$. Let $\mathscr{B} = L^2(\mathscr{O})$ and define the second-order differential operator

$$Au = \Delta u, \ \ \forall u \in D(A) = W^{2,p}(\mathscr{O}) \cap W_0^{1,p}(\mathscr{O}).$$

It can be shown that A is sectorial.

Example 2.22. Let $\mathscr{O} \subset \mathbb{R}^N$ be a bounded open subset whose boundary $\partial\mathscr{O}$ is of class C^2. Let $n(x)$ denote the outer normal to $\mathscr{O}$ for each $x \in \partial\mathscr{O}$.

Consider the differential operator defined by

$$A_0u(x) = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u(x)$$

where the coefficients a_{ij} and b_i and c are real, bounded, and continuous on $\overline{\mathscr{O}}$. Moreover, we suppose that for each $x \in \overline{\mathscr{O}}$, the matrix $[a_{ij}(x)]$ is symmetric and strictly positive definite, that is,

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \omega|\xi|^2 \text{ for all } x \in \overline{\mathscr{O}}, \ \ \xi \in \mathbb{R}^N.$$

Theorem 2.8. *(S. Agmon [5] and Lunardi et al. [131]) Let $p > 1$.*

(i) Let $A_p : W^{2,p}(\mathbb{R}^N) \mapsto L^p(\mathbb{R}^N)$ be the linear operator defined by $A_pu = A_0u$. Then the operator A_p is sectorial in $L^p(\mathbb{R}^N)$ and the domain $D(A_p)$ is dense in $L^p(\mathbb{R}^N)$.

(ii) Let A_0 be defined as above and let A_p be the linear operator defined by

$$D(A_p) = W^{2,p}(\mathscr{O}) \cap W_0^{1,p}(\mathscr{O}), \ \ A_pu = A_0u.$$

Then the linear operator A_p is sectorial in $L^p(\Omega)$. Moreover, $D(A_p)$ is dense in $L^p(\mathscr{O})$.

(iii) Let A_0 be defined as above and let A_p be the linear operator defined by

$$D(A_p) = \{u \in W^{2,p}(\mathscr{O}) : Bu_{|_{\partial\mathscr{O}}} = 0\}, \ \ A_pu = A_0u, \ \ u \in D(A_p)$$

where

$$Bu(x) = b_0u(x) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i}$$

with the coefficients b_i ($i = 1, \ldots, N$) are in $C^1(\overline{\mathscr{O}})$ and the condition

$$\sum_{i=1}^{N} b_i(x)n_i(x) \neq 0 \ \ x \in \partial\mathscr{O}$$

holds. Then A_p is sectorial in $L^p(\mathscr{O})$ and $D(A_p)$ is dense in $L^p(\mathscr{O})$.

2.4 Semigroups of Linear Operators

2.4.1 Basic Definitions

Definition 2.16. Let $(\mathscr{B}, \|\cdot\|)$ be a Banach space. The family of bounded operators $(T(t))_{t\in\mathbb{R}^+} : \mathscr{B} \to \mathscr{B}$ is said to be a semigroup or one–parameter semigroup if the following statements hold true:

(i) $T(0) = I$; and

(ii) $T(t+s) = T(t)T(s)$ for all $s,t \geq 0$.
Moreover if
(iii) $\lim_{t\searrow 0} \|T(t) - I\| = 0$, then the semigroup $T(t)$ is said to be uniformly continuous.

Remark 2.7. If $(T(t))_{t\in\mathbb{R}^+} : \mathscr{B} \to \mathscr{B}$ is a semigroup of bounded linear operator, one can associate with it an operator $(D(A), A)$ called the infinitesimal generator of the semigroup defined by

$$D(A) := \left\{ u \in \mathscr{B} : \lim_{t\searrow 0} \frac{T(t)u - u}{t} \text{ exists} \right\}, \tag{2.17}$$

and

$$Au := \lim_{t\searrow 0} \frac{T(t)u - u}{t}, \text{ for every } u \in D(A). \tag{2.18}$$

Remark 2.8. An operator A is the infinitesimal generator of a uniformly continuous semigroup of bounded linear operators $(T(t))_{t\in\mathbb{R}^+}$ if and only if A is bounded. In that event, it can be shown that $T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$.

Definition 2.17. A semigroup of bounded linear operators $(T(t))_{t\in\mathbb{R}^+} : \mathscr{B} \mapsto \mathscr{B}$ is said to be a strongly continuous semigroup of bounded linear operators (or c_0-semigroup) if $\lim_{t\searrow 0} \|T(t)x - x\| = 0$ for each $x \in \mathscr{B}$.

Example 2.23. Suppose that $\mathscr{B} = (BUC(\mathbb{R}), \|\cdot\|_\infty)$ is the Banach space of bounded uniformly continuous functions on the real number line equipped with the sup norm. Define

$$(S(t)\phi)(\sigma) = \phi(t+\sigma), \ \forall \phi \in BUC(\mathbb{R}).$$

Then $(S(t))_{t\in\mathbb{R}}$ is a c_0-semigroup with $\|S(t)\| \leq 1$ for each $t \in [0,\infty)$. Moreover, its infinitesimal generator A is defined by

$$D(A) = H^1(\mathbb{R}), \text{ and } A\varphi = \varphi', \ \forall \varphi \in H^1(\mathbb{R}),$$

where $H^1(\mathbb{R})$ is the Sobolev space.

Example 2.24. Let $1 \leq p < \infty$ and let $\mathscr{B} = L^p(\mathbb{R})$ equipped with its natural norm $\|\cdot\|_p$. Define $(S(0))u(x) = u(x)$ for all $x \in \mathbb{R}$, and

$$(S(t))u(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-|x-y|^2}{4t}} u(y)dy, \ t > 0, \ x \in \mathbb{R}.$$

Then $S(t)$ is a c_0-semigroup satisfying

$$\|S(t)u\|_p \leq \|u\|_p$$

and whose infinitesimal generator A_p is defined by

$$D(A_p) = W^{2,p}(\mathbb{R}), \;\; A_p u = u'', \;\; \text{for all} \;\; u \in D(A_p).$$

Example 2.25. This is a generalization of Example 2.24. Let $1 \leq p < \infty$ and let $\mathscr{B} = L^p(\mathbb{R}^N)$ (or $BC(\mathbb{R}^N, \mathbb{C})$ equipped with the sup norm) equipped with its natural norm $\|\cdot\|_p$. Define $(S(0))u(x) = u(x)$ for all $x \in \mathbb{R}^N$, and

$$(S(t))u(x) = \frac{1}{(4\pi t)^{N/2}} \int_{-\infty}^{\infty} e^{\frac{-\|x-y\|^2}{4t}} u(y)dy, \;\; t > 0, \;\; x \in \mathbb{R}.$$

Then $S(t)$ is a c_0-semigroup satisfying

$$\|S(t)u\|_p \leq \|u\|_p$$

and whose infinitesimal generator A_p is defined by

$$D(A_p) = W^{2,p}(\mathbb{R}^N), \;\; A_p u = \Delta u, \;\; \text{for all} \;\; u \in D(A_p).$$

2.4.2 Basic Properties of Semigroups

Theorem 2.9. *Let $(T(t))_{t \in \mathbb{R}^+} : \mathscr{B} \to \mathscr{B}$ be a semigroup of bounded linear operators, then*

(i) there are constants C, ζ such that $\|T(t)\| \leq C\, e^{\zeta t}$, $t \in \mathbb{R}^+$;

(ii) the infinitesimal generator A of the semigroup $T(t)$ is a densely defined closed operator;

(iii) the map $t \mapsto T(t)x$ which goes from $\mathbb{R}^+$ into $\mathscr{B}$ is continuous for every $x \in \mathscr{B}$;

(iv) the differential equation given by

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax,$$

holds for every $x \in D(A)$;

(v) for every $x \in \mathscr{B}$, then $T(t)x = \lim_{\lambda \searrow 0} (\exp(tA_\lambda))x$, with

$$A_\lambda x := \frac{T(\lambda)x - x}{\lambda},$$

where the above convergence is uniform on every compact subset of $\mathbb{R}^+$; and

(vi) if $\lambda \in \mathbb{C}$ with $\Re e\lambda > \zeta$, then the integral

$$R(\lambda, A)x := (\lambda I - A)^{-1} x = \int_0^{\infty} e^{-\zeta t}\, T(t)x\, dt,$$

defines a bounded linear operator $R(\lambda, A)$ on $\mathscr{B}$ whose range is $D(A)$ and

$$(\lambda I - A)\, R(\lambda, A) = R(\lambda, A)(\lambda I - A) = I.$$

Proof. For the proof, we refer the reader to the book by Pazy [153].

Remark 2.9. In (i) above if $\zeta = 0$, then the corresponding semigroup is uniformly bounded. Moreover, if $C = 1$, then $(T(t))_{t\in\mathbb{R}^+}$ is said to be a c_0-semigroup of contractions.

Theorem 2.10. (Hille–Yosida) *Let $A : D(A) \to \mathscr{B}$ be an unbounded linear operator in a Banach space $\mathbb{H}$. Then A is the infinitesimal generator of a c_0-semigroup of contractions $(T(t))_{t\in\mathbb{R}^+}$ if and only if:*

(i) A is a densely defined closed operator; and

(ii) the resolvent $\rho(A)$ of A contains $\mathbb{R}^+$ and

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}, \quad \forall\lambda > 0. \tag{2.19}$$

Proof. For the proof, we refer the reader to the book by Pazy [153].

Definition 2.18. Let $\mathscr{B}$ be a Banach space. The family of bounded operators $(T(t))_{t\in\mathbb{R}} : \mathscr{B} \to \mathscr{B}$ is said to be a c_0-*group* if the following statements hold true:

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s)$ for every $s,t \in \mathbb{R}$,

(iii) $\lim\limits_{t\to 0} \Big\|T(t)x - x\Big\| = 0$ for $x \in \mathscr{B}$.

Remark 2.10. As for semigroups of bounded linear operators, for a given c_0-group $(T(t))_{\mathbb{R}}$ one can associate with it an infinitesimal generator A defined as in (3.3) and (3.4).

We have

Theorem 2.11. *Let $A : D(A) \to \mathscr{B}$ be a linear operator on $\mathscr{B}$. Then A is the infinitesimal generator of a c_0-group of bounded linear operators $(T(t))_{t\in\mathbb{R}}$ satisfying $\|T(t)\| \leq C\, e^{\zeta|t|}$ if and only if:*

(i) A is a densely defined closed operator; and

(ii) every $\lambda \in \mathbb{R}$ such that $|\lambda| \geq \zeta$ is in $\rho(A)$ and that for such a λ, the following holds:

$$\Big\|(\lambda I - A)^{-n}\Big\| \leq \frac{C}{(|\lambda| - \zeta)^n}. \tag{2.20}$$

Proof. For the proof, we refer the reader to the book by Pazy [153].

2.4.3 Analytic Semigroups

Definition 2.19. A semigroup $T(t)$ on $\mathscr{B}$ is called analytic whenever $t \mapsto T(t)$ is analytic in $(0,\infty)$ with values in $B(\mathscr{B})$.

Let us mention that if $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ is a sectorial operator with constants $\zeta \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, and $M > 0$, then one can construct an analytic semigroup $T(t)$ associated to A by the means of the Dunford integral as follows (see Lunardi [129]):

$$T(t) = \frac{1}{2\pi i} \int_{\zeta + \Gamma_{r,s}} e^{t\lambda} R(\lambda, A) d\lambda, \quad \forall t > 0, \tag{2.21}$$

where $r > 0$, $\pi/2 < s < \theta$, and $\Gamma_{r,s}$ is the curve of the complex plane given by

$$\Big\{\lambda \in \mathbb{C} : \big|\arg \lambda\big| = s, \big|\lambda\big| \geq r\Big\} \cup \Big\{\lambda \in \mathbb{C} : \big|\arg \lambda\big| \leq s, \big|\lambda\big| = r\Big\},$$

that is oriented counterclockwise.

We have

Proposition 2.13. *[129] Let A be a sectorial operator and let $T(t)$ be the analytic semigroup given in (2.21). Then the following hold:*

(i) $T(t)u \in D(A^k)$ for all $t > 0$, $u \in \mathscr{B}$, $n \in \mathbb{N}$. If $D(A^n)$, then

$$A^n T(t)u = T(t)A^n u, \;\; t \geq 0;$$

(ii) there exist constants $M_0, M_1, \ldots$ such that

$$\Big\|T(t)\Big\| \leq M_0 e^{\zeta t}, \;\; t > 0, \;\; and$$

$$\Big\|t^n (A - \zeta I)^n T(t)\Big\| \leq M_n e^{\zeta t}, \;\; t > 0; \;\; and$$

(iii) the mapping $t \to T(t)$ belongs to $C^\infty((0,\infty), B(\mathbb{H}))$ and

$$\frac{d^n}{dt^n} T(t) = A^n T(t), \;\; t > 0, \, \forall n \in \mathbb{N}.$$

Conversely, the next proposition characterizes analytic semigroups in terms of sectorial operators.

Proposition 2.14. *[129] Let $(T(t))_{t>0}$ be a family of bounded linear operators on $\mathscr{B}$ such that $t \mapsto T(t)$ is differentiable, and*

(i) $T(t+s) = T(t)T(s)$ for all $t, s > 0$;

(ii) there exist $\zeta \in \mathbb{R}$, $M_0, M_1 > 0$ such that

$$\Big\|T(t)\Big\| \leq M_0 e^{\zeta t}, \quad \Big\|tT'(t)\Big\| \leq M_1 e^{\zeta t}, \quad \forall t > 0;$$

(iii) either (a) there exists $t > 0$ such that $T(t)$ is one-to-one, or (b) for every $x \in \mathscr{B}$, $s - \lim_{t \to 0} T(t)x = x$.

Then $t \mapsto T(t)$ is analytic in $(0,\infty)$ with values in $B(\mathscr{B})$, and there exists a unique sectorial operator $A : D(A) \subset \mathbb{H} \to \mathscr{B}$ such that $(T(t))_{t\geq 0}$ is the semigroup associated with A.

Proof. For the proof, we refer the reader to the book by Lunardi [129].

2.5 Intermediate Spaces

2.5.1 Fractional Powers of Sectorial Operators

Let A be a sectorial linear operator on $\mathscr{B}$ whose associated analytic semigroup $T(t)$ satisfies the following: For all $t > 0$,

$$\|T(t)\| \leq M_0 e^{-\omega t}, \quad \|tAT(t)\| \leq M_1 e^{-\omega t},$$

where $M_0, M_1, \omega > 0$.

For each $\alpha > 0$ one defines the fractional powers of $-A$ implicitly by

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t) dt, \tag{2.22}$$

where Γ is defined by $\Gamma(x) := \displaystyle\int_0^{+\infty} e^{-xt} t^{x-1} dt$ for each $x > 0$.

Lemma 2.1. *For all $\alpha, \beta > 0$, the following hold:*
(i) $(-A)^{-\alpha}(-A)^{-\beta} = A^{-(\alpha+\beta)}$.
(ii) $\lim\limits_{\alpha \to 0} (-A)^{-\alpha} = I$ *in the strong operator topology.*

Proof.

$$\begin{aligned}
(-A)^{-\alpha}(-A)^{-\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{+\infty} \int_0^{+\infty} t^{\alpha-1} s^{\beta-1} T(t)T(s) dt ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{+\infty} t^{\alpha-1} \int_t^{+\infty} (u-t)^{\beta-1} T(u) du dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{+\infty} \int_0^{u} t^{\alpha-1} (u-t)^{\beta-1} dt T(u) du \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{1} v^{\alpha-1} (1-v)^{\beta-1} dv \int_0^{\infty} u^{\alpha+\beta-1} T(u) du \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_0^{+\infty} u^{\alpha+\beta-1} T(u) du \\
&= (-A)^{-\alpha-\beta}.
\end{aligned}$$

It remains to prove that $(-A)^{-\alpha} \to I$ as $\alpha \to 0$. Since $(-A)^{-\alpha}$ is one-to-one, if $v \in D(A)$, there exists $u \in \mathbb{H}$ such that $v = (-A)^{-\alpha} u$. Thus $(-A)^{-\alpha} v - v = (-A)^{-1-\alpha} u - (-A)^{-1} u \to 0$ as $\alpha \to 0$ by the fact that $(-A)^{-\alpha}$ is continuous with respect to uniform operator norm.

Remark 2.11. (i) Let $\alpha \in (0,1)$. Using the fact that

$$(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)dt,$$

the formula (2.22) can be rewritten as

$$(-A)^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^{+\infty} \lambda^{-\alpha} (\lambda I - A)^{-1} dt. \tag{2.23}$$

(ii) The operator $(-A)^{-\alpha}$ is one-to-one, and hence has an inverse, which obviously is $(-A)^{\alpha}$. The operator $(-A)^{\alpha}$ is closed with domain $D((-A)^{\alpha}) = R((-A)^{-\alpha})$. The operators $(-A)^{\alpha}$ are called fractional powers of $-A$.

(iii) If $\alpha > \beta$, then $D((-A)^{\alpha}) \subset D((-A)^{\beta})$.

(iv) $D((-A)^{\alpha})$ is endowed with the norm $\|u\|_\alpha = \|(-A)^{\alpha} u\|$ for each $u \in D((-A)^{\alpha})$.

(v) $(-A)^{\alpha}$ commutes with $T(t)$ on $D(-A)^{\alpha})$ with

$$\|T(t)\|_{B(D(-A)^{\alpha}))} \leq M_0 e^{-\omega t}, \ \ t > 0.$$

Example 2.26. Let A be the operator given by $Au = -u''$ for all $u \in D(A)$ where the domain $D(A)$ is defined by

$$D(A) := \{u \in L^2([0,\pi]) : u'' \in L^2([0,\pi]),\ u(0) = u(\pi) = 0\}.$$

Clearly, the operator A has a discrete spectrum with eigenvalues of the form $n^2, n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by

$$z_n(\xi) := \sqrt{\frac{2}{\pi}} \sin(n\xi).$$

In addition to the above, the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for $L^2[0,\pi]$.

(b) The operator $-A$ is the infinitesimal generator of an analytic semigroup $R(t)$ which is compact for $t > 0$. The semigroup $R(t)$ is defined for $u \in L^2[0,\pi]$ by

$$R(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, z_n \rangle z_n.$$

(c) The operator A can be rewritten as

$$Au = \sum_{n=1}^{\infty} n^2 \langle u, z_n \rangle z_n$$

for every $u \in D(A)$.

Moreover, it is possible to define fractional powers of A. In particular,

(d) For $u \in L^2[0,\pi]$ and $\alpha \in (0,1)$,

$$A^{-\alpha}u = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \langle u, z_n \rangle z_n.$$

(e) The operator $A^{\alpha} : D(A^{\alpha}) \subseteq L^2[0,\pi] \mapsto L^2[0,\pi]$ given by

$$A^{\alpha}u = \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n, \quad \forall u \in D(A^{\alpha}),$$

where $D(A^{\alpha}) = \left\{ u \in L^2[0,\pi] : \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n \in L^2[0,\pi] \right\}$.

Clearly, for all $t \geq 0$ and $0 \neq u \in L^2[0,\pi]$,

$$\begin{aligned}
|R(t)u| &= |\sum_{n=1}^{\infty} e^{-n^2 t} \langle u, z_n \rangle z_n| \\
&\leq \sum_{n=1}^{\infty} e^{-t} |\langle u, z_n \rangle z_n| \\
&= e^{-t} \sum_{n=1}^{\infty} |\langle u, z_n \rangle z_n| \\
&\leq e^{-t} |u|
\end{aligned}$$

and hence $\|R(t)\|_{B(L^2[0,\pi])} \leq 1$ for all $t \geq 0$.

2.5.2 *The Spaces* $D_A(\alpha, p)$ *and* $D_A(\alpha)$

Let A be a sectorial linear operator on $\mathscr{B}$ whose associated analytic semigroup $T(t)$ satisfies the following: For all $t > 0$,

$$\|T(t)\| \leq M_0 e^{-\omega t}, \quad \|tAT(t)\| \leq M_1 e^{-\omega t},$$

where $M_0, M_1, \omega > 0$.

Definition 2.20. Let $\alpha \in (0,1)$. A Banach space $(\mathscr{B}_{\alpha}, \|\cdot\|_{\alpha})$ is called an intermediate space between $\mathscr{B}$ and $D(A)$, or a space of class $\mathscr{J}_{\alpha}$, if $D(A) \subset \mathscr{B}_{\alpha} \subset \mathscr{B}$ and there is a constant $C > 0$ such that

$$\|u\|_{\alpha} \leq C \|u\|^{1-\alpha} \|u\|_A^{\alpha}, \qquad u \in D(A). \tag{2.24}$$

Concrete examples of $\mathscr{B}_{\alpha}$ include $D((-A)^{\alpha})$ for $\alpha \in (0,1)$, the domains of the fractional powers of $-A$, the real interpolation spaces $D_A(\alpha, \infty)$, $\alpha \in (0,1)$, defined as follows:

Definition 2.21. Let $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ be a sectorial operator and let $\alpha \in (0,1)$. Define

$$D_A(\alpha,\infty) := \left\{ u \in \mathscr{B} : [u]_\alpha = \sup_{0\leq t\leq 1} \left\| t^{1-\alpha} AT(t)u \right\| < \infty \right\}$$

equipped with the norm given by

$$\|u\|_{D(\alpha,\infty)} = \|u\| + [u]_\alpha \, .$$

One should point out that $D_A(\alpha,\infty)$ is characterized by the behavior of the quantity $t \mapsto \|t^{1-\alpha} AT(t)u\|$ near $t = 0$. Moreover, all the spaces $D_A(\alpha,\infty)$ are subspaces of $\overline{D(A)}$. Namely, the following embeddings hold with equivalent norms:

$$D(A) \subset D_A(\beta,\infty) \subset D_A(\alpha,\infty) \subset \overline{D(A)}$$

for all $0 < \alpha < \beta < 1$.

If $\alpha \in (0,1)$, it is not very hard to see that $D_A(\alpha,\infty)$ can be characterized as being the subspace of all $u \in \mathscr{B}$ such that

$$[[u]]_\alpha = \sup_{t\in(0,1]} t^{-\alpha} \|T(t)u - u\| < \infty.$$

Furthermore, the norm defined by $u \mapsto \|u\| + [[u]]_\alpha$ is equivalent to the natural norm of $D_A(\alpha,\infty)$.

More generally, we define $D_A(\alpha, p)$ for $\alpha \in (0,1)$ and $1 \leq p \leq \infty$ as follows:

Definition 2.22. Let $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ be a sectorial operator. Define the classes of intermediate spaces $D_A(\alpha, p)$ and $D_A(\alpha)$ between $\mathscr{B}$ and $D(A)$ (for $\alpha \in (0,1)$ and $1 \leq p \leq \infty$) by

$$D_A(\alpha, p) := \left\{ u \in \mathscr{B} : t \mapsto v(t) = \left\| t^{1-\alpha-1/p} AT(t) \right\| \in L^p(0,1) \right\}$$

equipped with the norm given by

$$\|u\|_{D(\alpha,p)} = \|u\| + [u]_{D(\alpha,p)} = \|u\| + \|v\|_{L^p(0,1)}\, ;$$

and

$$D_A(\alpha) = \left\{ u \in D(\alpha,\infty) : \lim_{t\to 0} t^{1-\alpha} AT(t)u = 0 \right\}.$$

Proposition 2.15. *For* $\alpha \in (0,1)$ *and* $1 \leq p \leq \infty$ *and for* $(\alpha, p) = (1,\infty)$, *then*

$$D_A(\alpha, p) = (\mathscr{B}, D(A))_{\alpha,p}$$

with equivalent norms. Moreover, for $0 < \alpha < 1$, *then*

$$D_A(\alpha) = (\mathscr{B}, D(A))_\alpha .$$

Proof. The proof of Proposition 2.15 is too technical and so we refer the reader to Lunardi [129].

Proposition 2.16. *[129] For $\alpha \in (0,1)$, then*

$$D_A(\alpha,1) \subset D((-A)^\alpha) \subset (\mathscr{B}, D(A))_{\alpha,p}.$$

Proof. First of all, note that $D((-A)^\alpha)$ belongs to the class J_α. Notice that for each $u \in D(A)$, $(-A)^\alpha u = (-A)^{-(1-\alpha)}(-Au)$ and hence for each $\lambda > 0$,

$$\begin{aligned} \|(-A)^\alpha u\| &= \frac{1}{\Gamma(1-\alpha)} \Big\| \Big(\int_0^\lambda + \int_\lambda^\infty \Big) t^{-\alpha} A e^{tA} u dt \Big\| \\ &\leq \frac{1}{\Gamma(1-\alpha)} \Big(\frac{M_0}{1-\alpha} \|Au\| \lambda^{1-\alpha} + \frac{M_1}{\alpha} \|u\| \lambda^{-\alpha} \Big). \end{aligned}$$

Letting $\lambda = \dfrac{\|u\|}{\|Au\|}$ it follows that

$$\|(-A)^\alpha u\| \leq c \|Au\|^\alpha \|u\|^{1-\alpha}.$$

It remains to prove that $D(-A)^\alpha)$ is continuously embedded in $D_A(\alpha,\infty)$. For that let $u \in D((-A)^\alpha)$ and let $v = (-A)^\alpha u$. So for $0 < \xi \leq 1$, we have

$$\begin{aligned} \|\xi^{1-\alpha} A e^{\xi A} u\| &= \|\xi^{1-\alpha} A e^{\xi A} (-A)^\alpha v\| \\ &\leq \frac{\xi^{1-\alpha}}{\Gamma(\alpha)} \Big\| \int_0^\infty t^{\alpha-1} A e^{(\xi+t)A} v dt \Big\| \\ &\leq \frac{M_1 \xi^{1-\alpha}}{\Gamma(\alpha)} \int_0^\infty \frac{\xi^{\alpha-1}}{\xi+t} dt \|v\| \\ &\leq \frac{M_1}{\Gamma(\alpha)} \int_0^\infty \frac{s^{\alpha-1}}{1+s} ds \|(-A)^\alpha u\| \end{aligned}$$

and hence $u \in D_A(\alpha,\infty)$.

Using $D_A(\alpha,\infty)$ we can define $D_A(k+\alpha,\infty)$ as follows:

Definition 2.23. Let $A : D(A) \subset \mathscr{B} \to \mathscr{B}$ be a sectorial operator. For any $k \in \mathbb{N}$ and any $\alpha \in (0,1)$, we define

$$D_A(\alpha+k,\infty) := \Big\{ u \in D(A^k) : A^k u \in D_A(\alpha,\infty) \Big\}$$

equipped with the norm given by

$$\|u\|_{D(\alpha+k,\infty)} = \|u\|_{D(A^k)} + \Big[A^k u \Big]_\alpha.$$

Let A_α denote the part of A in $D_A(\alpha,\infty)$. It can be shown that $A_\alpha : D_A(1+\alpha,\infty) \mapsto D_A(\alpha,\infty)$ with $A_\alpha u = Au$ is sectorial. Moreover, $\rho(A) \subset \rho(A_\alpha)$. Furthermore, the restriction of $R(\lambda,A)$ to $D_A(\alpha,\infty)$ is exactly $R(\lambda,A_\alpha)$ and

$$\|R(\lambda, A_\alpha)\|_{B(D_A(\alpha,\infty))} \leq \|R(\lambda, A)\|$$

for all $\lambda \in \rho(A)$.

Example 2.27. Let A be a realization of the Laplacian in $\mathscr{B} = BC(\mathbb{R}^N, \mathbb{C})$. Then for all $\alpha \in (0,1)$ and $\alpha \neq 1/2$, then

$$D_A(\alpha, \infty) = C_b^{2\alpha}(\mathbb{R}^N) \tag{2.25}$$

and

$$D_A(1+\alpha, \infty) = C_b^{2+2\alpha}(\mathbb{R}^N) \tag{2.26}$$

with equivalent norms.

For more on those spaces and related issues we refer the reader to the landmark book by Lunardi [129].

2.5.3 Hyperbolic Semigroups

Definition 2.24. Let A be a sectorial operator on $\mathscr{B}$ and let $(T(t))_{t\geq 0}$ be the analytic semigroup associated to it. The semigroup $(T(t))_{t\geq 0}$ is said to be hyperbolic if there exist a projection P and constants $M, \delta > 0$ such that each $T(t)$ commutes with P, $N(P)$ is invariant with respect to $T(t)$, $T(t) : R(Q) \mapsto R(Q)$ is invertible, and

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\| \qquad \text{for } t \geq 0, \tag{2.27}$$

$$\|T(t)Qx\| \leq Me^{\delta t}\|x\| \qquad \text{for } \ t \leq 0, \tag{2.28}$$

where $Q := I - P$ and $T(t) := (T(-t))^{-1}$ for $t < 0$.

Recall that an analytic semigroup $(T(t))_{t\geq 0}$ is hyperbolic if and only if (see [70])

$$\sigma(A) \cap i\mathbb{R} = \emptyset. \tag{2.29}$$

For the hyperbolic analytic semigroup $T(t)$, we can easily check that estimations similar to (2.27) and (2.28) hold also with norms $\|\cdot\|_\alpha$ (see Definition 2.20). In fact, as the part of A in $R(Q)$ is bounded, it follows from the inequality (2.28) that

$$\|AT(t)Qx\| \leq c'e^{\delta t}\|x\| \qquad \text{for } t \leq 0.$$

In view of the above, there exists a constant $c(\alpha) > 0$ such that

$$\|T(t)Qx\|_\alpha \leq c(\alpha)e^{\delta t}\|x\| \qquad \text{for } t \leq 0. \tag{2.30}$$

Similarly,

$$\|T(t)Px\|_\alpha \le \|T(1)\|_{B(\mathscr{B},\mathscr{B}_\alpha)}\|T(t-1)Px\| \qquad \text{for } t \ge 1,$$

and then from (2.27), we obtain

$$\|T(t)Px\|_\alpha \le M' e^{-\delta t}\|x\|, \qquad t \ge 1,$$

where M' depends on α.

Clearly,

$$\|T(t)Px\|_\alpha \le M'' t^{-\alpha}\|x\|,$$

and hence there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \le M(\alpha) t^{-\alpha} e^{-\gamma t}\|x\| \qquad \text{for } t > 0. \tag{2.31}$$

We need the next lemma, which will be very crucial for our computations.

Lemma 2.2. *(Diagana [52]) Let* $0 < \alpha, \beta < 1$. *Then*

$$\|AT(t)Qx\|_\alpha \le ce^{\delta t}\|x\|_\beta \quad \textit{for } t \le 0, \tag{2.32}$$

$$\|AT(t)Px\|_\alpha \le ct^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta \quad \textit{for } t > 0. \tag{2.33}$$

Proof. As for (2.30), the fact that the part of A in $R(Q)$ is bounded yields

$$\|AT(t)Qx\| \le ce^{\delta t}\|x\|_\beta, \quad \|A^2T(t)Qx\| \le ce^{\delta t}\|x\|_\beta \quad \text{for } t \le 0, \tag{2.34}$$

since $\mathscr{B}_\beta \hookrightarrow \mathscr{B}$. Hence, from (2.24) there is a constant $c(\alpha) > 0$ such that

$$\|AT(t)Qx\|_\alpha \le c(\alpha)e^{\delta t}\|x\|_\beta \quad \text{for } t \le 0. \tag{2.35}$$

Furthermore,

$$\|AT(t)Px\|_\alpha \le \|AT(1)\|_{B(\mathscr{B},\mathscr{B}_\alpha)}\|T(t-1)Px\| \tag{2.36}$$

$$\le ce^{-\delta t}\|x\|_\beta \quad \text{for } t \ge 1. \tag{2.37}$$

Now for $t \in (0,1]$, by Proposition 2.13 (ii) and (2.24), one has

$$\|AT(t)Px\|_\alpha \le ct^{-\alpha-1}\|x\|,$$

and

$$\|AT(t)Px\|_\alpha \le ct^{-\alpha}\|Ax\|,$$

for each $x \in D(A)$. Thus, by the Reiteration Theorem (see [129]), it follows that

$$\|AT(t)Px\|_\alpha \le ct^{\beta-\alpha-1}\|x\|_\beta$$

for every $x \in \mathscr{B}_\beta$ and $0 < \beta < 1$, and hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta \qquad \text{for } t > 0.$$

2.6 Evolution Families and Their Properties

2.6.1 Evolution Families

Let $\{A(t) : t \in \mathbb{R}\}$ be a family of closed linear operators on $\mathscr{B}$ with domain $D(A(t))$ (possibly not densely defined), which depends on $t \in \mathbb{R}$.

Definition 2.25. A family of linear operators

$$\{U(t,s) : t,s \in \mathbb{R} \text{ such that } t \geq s\}$$

on $\mathscr{B}$ associated with $A(t)$ such that $U(t,s)\mathscr{B} \subset D(A(t))$ for all $t,s \in \mathbb{R}$ with $t \geq s$, and

(a) $U(t,s)U(s,r) = U(t,r)$ for $t,s,r \in \mathbb{R}$ such that $t \geq s \geq r$;
(b) $U(t,t) = I$ for $t \in \mathbb{R}$;
(c) $(t,s) \mapsto U(t,s) \in B(\mathscr{B})$ is continuous for $t > s$; and
d) $U(\cdot,s) \in C^1((s,\infty),B(\mathscr{B}))$, $\dfrac{\partial U}{\partial t}(t,s) = A(t)U(t,s)$
is callled an evolution family.

For a given family of closed linear operators $\{A(t) : t \in \mathbb{R}\}$ on $\mathscr{B}$, the existence of an evolution family associated with it is not always guaranteed. However, if the family $A(t)$ satisfies the so-called Acquistapace–Terreni conditions, that is:
(AT) There exists $\lambda_0 \geq 0$ such that the linear operators $\{A(t) : t \in \mathbb{R}\}$ satisfy

$$\Sigma_\phi \cup \{0\} \subseteq \rho(A(t)-\lambda_0) \ni \lambda, \quad \|R(\lambda,A(t)-\lambda_0)\| \leq \frac{K}{1+|\lambda|} \tag{2.38}$$

and

$$\|(A(t)-\lambda_0)R(\lambda_0,A(t)-\lambda_0)\,[R(\lambda_0,A(t))-R(\lambda_0,A(s))]\,\| \leq L|t-s|^\mu\,|\lambda|^{-\nu} \tag{2.39}$$

for $t,s \in \mathbb{R}$, $\lambda \in \Sigma_\phi := \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg\lambda| \leq \phi\}$, and the constants $\phi \in (\frac{\pi}{2},\pi)$, $L,K \geq 0$, and $\mu,\nu \in (0,1]$ with $\mu+\nu > 1$, then the family of linear operators $A(t)$ has an evolution family associated to it. Moreover, the following hold:

(e)

$$\|A(t)^k U(t,s)\| \leq C\,(t-s)^{-k} \tag{2.40}$$

for $0 < t-s \leq 1$, $k = 0,1$; and

(f) $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

Remark 2.12. (i) In the particular case of a constant domain $D(A(t))$, one can replace assumption (2.39) (see for instance [153]) with the following:
(AT)$'$ There exist constants L and $0 < \mu \leq 1$ such that

$$\| (A(t) - A(s)) R(\lambda_0, A(r)) \| \leq L|t-s|^{\mu}, \ s,t,r \in \mathbb{R}. \tag{2.41}$$

(ii)The conditions (AT) were introduced in the literature by Acquistapace–Terreni in [2, 3] for $\lambda_0 = 0$.

Definition 2.26. An evolution family $\Big\{U(t,s) : t \geq s \ \text{ with } \ t,s \in \mathbb{R}\Big\} \subset B(\mathscr{B})$ is said to have an *exponential dichotomy* (*or is hyperbolic*) if there are projections $P(t)$ ($t \in \mathbb{R}$) that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(i) $U(t,s)P(s) = P(t)U(t,s)$;

(ii) the restriction $U_Q(t,s) : Q(s)\mathscr{B} \to Q(t)\mathscr{B}$ of $U(t,s)$ is invertible (we then set $U_Q(s,t) := U_Q(t,s)^{-1}$); and

(iii) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t,s \in \mathbb{R}$.

Here and throughout the rest of the book, for any projection P we set $Q = I - P$.

We recall that the following conditions are sufficient for an evolution family $\Big\{U(t,s) : t \geq s \ \text{ with } \ t,s \in \mathbb{R}\Big\}$ associated with $A(\cdot)$ to have exponential dichotomy:

(E_1) Let $(A(t), D(t))_{t \in \mathbb{R}}$ be generators of analytic semigroups on $\mathscr{B}$ of the same type. Suppose that $D(A(t)) \equiv D(A(0))$, $A(t)$ is invertible,

$$\sup_{t,s \in \mathbb{R}} \|A(t)A(s)^{-1}\|$$

is finite, and

$$\|A(t)A(s)^{-1} - I\| \leq L_0 |t-s|^{\mu}$$

for $t,s \in \mathbb{R}$ and constants $L_0 \geq 0$ and $0 < \mu \leq 1$.

(E_2) The semigroups $(e^{\tau A(t)})_{\tau \geq 0}$, $t \in \mathbb{R}$, are hyperbolic with projection P_t and constants $N, \delta > 0$. Moreover, let

$$\|A(t)e^{\tau A(t)} P_t\| \leq \psi(\tau)$$

and

$$\|A(t)e^{\tau A_Q(t)} Q_t\| \leq \psi(-\tau)$$

for $\tau > 0$ and a function ψ such that $\mathbb{R} \ni s \mapsto \varphi(s) := |s|^{\mu}\psi(s)$ is integrable with $L_0\|\varphi\|_{L^1(\mathbb{R})} < 1$.

2.6.2 Estimates for $U(t,s)$

We need to prove some estimates related to $U(t,s)$. For that, we introduce the interpolation spaces for $A(t)$. We refer the reader to [70], and [129] for proofs and further information on these spaces.

Let A be a sectorial operator on $\mathscr{B}$ and let $\alpha \in (0,1)$.

Define the real interpolation space

$$\mathscr{B}_\alpha^A := \left\{ x \in \mathscr{B} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha (A-\zeta)R(r,A-\zeta)x\| < \infty \right\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathscr{B}_0^A := \mathscr{B},\ \|x\|_0^A := \|x\|,\ \ \mathscr{B}_1^A := D(A),\ \text{ and }\ \|x\|_1^A := \|(\zeta - A)x\|.$$

We also need the closed subspace $\hat{\mathscr{B}}^A := \overline{D(A)}$ of $\mathscr{B}$. In particular, we will frequently be using the following continuous embedding:

$$D(A) \hookrightarrow \mathscr{B}_\beta^A \hookrightarrow D((\zeta - A)^\alpha) \hookrightarrow \mathscr{B}_\alpha^A \hookrightarrow \hat{\mathscr{B}}^A \subset \mathscr{B}, \tag{2.42}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathscr{B}_\alpha^A$ and $\mathscr{B}$. However, we have the following continuous injection:

$$\mathscr{B}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{2.43}$$

for $0 < \alpha < \beta < 1$.

Given the operators $A(t)$ for $t \in \mathbb{R}$, satisfying (AT), we set

$$\mathscr{B}_\alpha^t := \mathscr{B}_\alpha^{A(t)}, \quad \hat{\mathscr{B}}^t := \hat{\mathscr{B}}^{A(t)}$$

for $0 \le \alpha \le 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in (2.42) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathscr{J}_\alpha$ and hence there is a constant $l(\alpha)$ such that

$$\|y\|_\alpha^t \le l(\alpha)\|y\|^{1-\alpha}\|A(t)y\|^\alpha, \quad y \in D(A(t)). \tag{2.44}$$

We have the following fundamental estimates for the evolution family $U(t,s)$:

Proposition 2.17. *[14, Baroun, Boulite, Diagana, and Maniar] For $x \in \mathscr{B}$, $0 \le \alpha \le 1$ and $t > s$, the following hold:*

(i) There is a constant $c(\alpha)$, such that

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.45}$$

(ii) There is a constant $m(\alpha)$, such that

$$\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s \leq m(\alpha)e^{-\delta(t-s)}\|x\|. \tag{2.46}$$

Proof. (i) Using (2.44) we obtain

$$\begin{aligned}
\|U(t,s)P(s)x\|_\alpha^t &\leq c(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,s)P(s)x\|^\alpha \\
&\leq c(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,t-1)U(t-1,s)P(s)x\|^\alpha \\
&\leq l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,t-1)\|^\alpha\|U(t-1,s)P(s)x\|^\alpha \\
&\leq l(\alpha)N'\, e^{-\delta(t-s)(1-\alpha)}e^{-\delta(t-s-1)\alpha}\|x\| \\
&\leq c'(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)}\|x\|
\end{aligned}$$

for $t-s\geq 1$ and $x\in\mathscr{B}$.

Since $(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)}\to 0$ as $t\to\infty$ it easily follows that

$$\|U(t,s)P(s)x\|_\alpha^t \leq c_1(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\|.$$

If $0<t-s\leq 1$, we have

$$\begin{aligned}
\|U(t,s)P(s)x\|_\alpha^t &\leq l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,s)P(s)x\|^\alpha \\
&\leq l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,\frac{t+s}{2})U(\frac{t+s}{2},s)P(s)x\|^\alpha \\
&\leq l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,\frac{t+s}{2})\|^\alpha\|U(\frac{t+s}{2},s)P(s)x\|^\alpha \\
&\leq l(\alpha)Ne^{-\delta(t-s)(1-\alpha)}2^\alpha(t-s)^{-\alpha}e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\
&\leq l(\alpha)Ne^{-\frac{\delta}{2}(t-s)(1-\alpha)}2^\alpha(t-s)^{-\alpha}e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\
&\leq c_2(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|,
\end{aligned}$$

and hence

$$\|U(t,s)P(s)x\|_\alpha^t \leq c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\| \text{ for } t>s.$$

(ii)

$$\begin{aligned}
\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s &\leq l(\alpha)\|\widetilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)\widetilde{U}_Q(s,t)Q(t)x\|^\alpha \\
&\leq l(\alpha)\|\widetilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\widetilde{U}_Q(s,t)Q(t)x\|^\alpha \\
&\leq l(\alpha)\|\widetilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\|^\alpha\|\widetilde{U}_Q(s,t)Q(t)x\|^\alpha \\
&\leq l(\alpha)Ne^{-\delta(t-s)(1-\alpha)}\|A(s)Q(s)\|^\alpha e^{-\delta(t-s)\alpha}\|x\| \\
&\leq m(\alpha)e^{-\delta(t-s)}\|x\|.
\end{aligned}$$

In the last inequality we made use of the fact that $\|A(s)Q(s)\|\leq c$ for some constant $c\geq 0$, see e.g., [162, Proposition 3.18].

Remark 2.13. It should be mentioned that if $U(t,s)$ is exponentially stable, then $P(t) = I$ and $Q(t) = I - P(t) = 0$ for all $t \in \mathbb{R}$. In that case, Eq. (2.45) still holds and can be rewritten as follows: for all $x \in \mathscr{B}$,

$$\|U(t,s)x\|_{\alpha}^{t} \leq c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.47}$$

We will need the following technical lemma in Chapters 5 and 6:

Lemma 2.3. *[55, Diagana] Let* $x \in \mathscr{B}$ *and let* $0 < \alpha < \beta < 1$ *with* $2\beta > \alpha + 1$. *Then for all* $t > s$, *there are constants* $r(\alpha,\beta), d(\beta) > 0$ *such that*

$$\|A(t)U(t,s)P(s)x\|_{\alpha} \leq r(\alpha,\beta)e^{-\frac{\delta}{4}(t-s)}(t-s)^{-\beta}\|x\|. \tag{2.48}$$

and

$$\left\|A(t)\widetilde{U}_Q(t,s)Q(s)x\right\|_{\beta} \leq d(\beta)e^{-\delta(s-t)}\left\|x\right\|, \qquad t \leq s. \tag{2.49}$$

Proof. Let $x \in \mathscr{B}$. First of all, note that $\|A(t)U(t,s)\|_{B(\mathscr{B},\mathscr{B}_\beta)} \leq K(t-s)^{-(1-\beta)}$ for all t,s such that $0 < t-s \leq 1$ and $\beta \in [0,1]$.

Suppose $t-s \geq 1$ and let $x \in \mathscr{B} \hookrightarrow \mathscr{B}_\alpha \hookrightarrow \mathscr{B}_\beta$.

$$\begin{aligned}
\|A(t)U(t,s)P(s)x\|_{\alpha} &= \|A(t)U(t,t-1)U(t-1,s)P(s)x\|_{\alpha} \\
&\leq \|A(t)U(t,t-1)\|_{B(\mathscr{B},\mathscr{B}_\alpha)}\|U(t-1,s)P(s)x\| \\
&\leq MKe^{\delta}e^{-\delta(t-s)}\|x\| \\
&= K_1e^{-\delta(t-s)}\|x\| \\
&= K_1e^{-\frac{3\delta}{4}(t-s)}(t-s)^{\beta}(t-s)^{-\beta}e^{-\frac{\delta}{4}(t-s)}\|x\|.
\end{aligned}$$

Now since $e^{-\frac{3\delta}{4}(t-s)}(t-s)^{\beta} \to 0$ as $t \to \infty$ it follows that there exists $c_4(\beta) > 0$ such that

$$\|A(t)U(t,s)P(s)x\|_{\beta} \leq c_4(\beta)(t-s)^{-\beta}e^{-\frac{\delta}{4}(t-s)}\|x\|.$$

Now, let $0 < t-s \leq 1$. Using Eq. (2.45) and the fact that $2\beta > \alpha + 1$, we obtain

$$\begin{aligned}
\|A(t)U(t,s)P(s)x\|_{\alpha} &= \|A(t)U(t,\frac{t+s}{2})U(\frac{t+s}{2},s)P(s)x\|_{\alpha} \\
&\leq \|A(t)U(t,\frac{t+s}{2})\|_{B(\mathscr{B},\mathscr{B}_\alpha)}\|U(\frac{t+s}{2},s)P(s)x\| \\
&\leq c'k_1\|A(t)U(t,\frac{t+s}{2})\|_{B(\mathscr{B},\mathscr{B}_\alpha)}\|U(\frac{t+s}{2},s)P(s)x\|_{\alpha} \\
&\leq c'k_1K\Big(\frac{t-s}{2}\Big)^{\beta-1}c(\alpha)\Big(\frac{t-s}{2}\Big)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\| \\
&= c_5(\alpha,\beta)(t-s)^{\beta-1-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\| \\
&\leq c_5(\alpha,\beta)(t-s)^{-\beta}e^{-\frac{\delta}{4}(t-s)}\|x\|.
\end{aligned}$$

In summary, there exists $r(\alpha,\beta) > 0$ such that

$$\|A(t)U(t,s)P(s)x\|_\alpha \le r(\alpha,\beta)(t-s)^{-\beta} e^{-\frac{\delta}{4}(t-s)}\|x\|$$

for all $t,s \in \mathbb{R}$ with $t \ge s$.

Let $x \in \mathscr{B}$. Since the restriction of $A(s)$ to $R(Q(s))$ is a bounded linear operator it follows that

$$\begin{aligned}
\left\|A(t)\widetilde{U}_Q(t,s)Q(s)x\right\|_\beta &= \left\|A(t)A(s)^{-1}A(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_\beta \\
&\le \left\|A(t)A(s)^{-1}\right\|_{B(\mathscr{B},\mathscr{B}_\beta)}\left\|A(s)\widetilde{U}_Q(t,s)Q(s)x\right\| \\
&\le c_1\left\|A(t)A(s)^{-1}\right\|_{B(\mathscr{B},\mathscr{B}_\beta)}\left\|A(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_\beta \\
&\le c_1c_0\left\|A(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_\beta \\
&\le \tilde{c}\left\|\widetilde{U}_Q(t,s)Q(s)x\right\|_\beta \\
&\le \tilde{c}m(\beta)e^{-\delta(s-t)}\left\|x\right\| \\
&= d(\beta)e^{-\delta(s-t)}\left\|x\right\|
\end{aligned}$$

for $t \le s$ by using Eq. (2.46).

We have also the following estimates due to Diagana [62]. Here, we still assume that the Acquistapace–Terreni conditions hold and that the evolution family $U(t,s)$ associated with $A(\cdot)$ has exponential dichotomy.

Lemma 2.4. *[62, Diagana] Suppose* $0 \in \rho(A(t))$ *for all* $t \in \mathbb{R}$ *such that*

$$\sup_{t,s\in\mathbb{R}} \|A(s)A^{-1}(t)\|_{B(\mathscr{B},\mathscr{B}_\alpha)} < c_0; \tag{2.50}$$

and that there exist $0 < \alpha < \beta < 1$ *with* $2\beta > \alpha + 1$ *such that*

$$\mathscr{B}^t_\alpha = \mathscr{B}_\alpha \ \text{ and } \ \mathscr{B}^t_\beta = \mathscr{B}_\beta$$

for all $t \in \mathbb{R}$, *with equivalent norms. Then, there exist two constants* $m(\alpha,\beta), n(\alpha,\beta) > 0$ *such that*

$$\|A(s)\widetilde{U}_Q(t,s)Q(s)x\|_\alpha \le m(\alpha,\beta)e^{-\delta(s-t)}\|x\|_\beta \quad \text{for } t \le s, \tag{2.51}$$

and

$$\|A(s)U(t,s)P(s)x\|_\alpha \le n(\alpha,\beta)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\|_\beta \quad \text{for } t > s. \tag{2.52}$$

Proof. Let $x \in \mathscr{B}_\beta$. Since the restriction of $A(s)$ to $R(Q(s))$ is a bounded linear operator it follows that

$$\begin{aligned}
\|A(s)\widetilde{U}_Q(t,s)Q(s)x\|_\alpha &\leq ck(\alpha)\,\|\widetilde{U}_Q(t,s)Q(s)x\|_\beta \\
&\leq ck(\alpha)m(\beta)e^{-\delta(s-t)}\,\|x\| \\
&\leq m(\alpha,\beta)e^{-\delta(s-t)}\|x\|_\beta
\end{aligned}$$

for $t \leq s$ by using (2.46).

Similarly, for each $x \in \mathscr{B}_\beta$, using (2.50), we obtain

$$\begin{aligned}
\|A(s)U(t,s)P(s)x\|_\alpha &= \|A(s)A(t)^{-1}A(t)U(t,s)P(s)x\|_\alpha \\
&\leq \|A(s)A(t)^{-1}\|_{B(\mathscr{B},\mathscr{B}_\alpha)}\|A(t)U(t,s)P(s)x\|_\alpha \\
&\leq c_0\|A(t)U(t,s)P(s)x\|_\alpha \\
&\leq c_0 r(\alpha,\beta)(t-s)^{-\beta}e^{-\frac{\delta}{4}(t-s)}\|x\| \\
&= n(\alpha,\beta)(t-s)^{-\beta}e^{-\frac{\delta}{4}(t-s)}\|x\|
\end{aligned}$$

for $t \geq s$.

2.7 Bibliographical Notes

For the classical theory of bounded linear operators, we follow, for the most part, essentially Conway [40], Diagana [51], Gohberg, Goldberg, and Kaashoek [78], Lax [115], Eidelman, Milman, and Tsolomitis [69], Naylar and Sell [146], Rudin [159], and Weidmann [176]. Our presentation related to unbounded linear operators, their spectral theory, and invariant and reducing subspaces for unbounded linear operators are taken from Diagana [51], Eidelman, Milman, and Tsolomitis [69], Locker [130], and Weidmann [176].

The part of this chapter devoted to semigroups was taken from Pazy [153]. However, the parts on sectorial operators, analytic semigroups, and intermediate spaces were taken from Lunardi [129]. The presentation on hyperbolic semigroups is due to Engel and Nagel [70].

Chapter 3
An Introduction to Stochastic Differential Equations

3.1 Fundamentals of Probability

In this section we review some basic concepts of probability theory and illustrate them with various examples. The review includes among other things the concepts of sample space, σ-field, events, probability measure, probability space, random variable, expectation, convergence of sequences of random variables, and conditional expectation.

3.1.1 Probability and Random Variables

The mathematical model for a random quantity is a random variable. Prior to giving a precise definition of this, we first recall some basic concepts from general probability theory.

Definition 3.1. A collection of subsets of a set Ω is called a *σ-field* or *σ-algebra*, denoted by $\mathscr{F}$, if it satisfies the following three properties:

(i) $\Omega \in \mathscr{F}$;

(ii) if $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$ ($\mathscr{F}$ is closed under complementation);

(iii) if $A_1, A_2, \ldots \in \mathscr{F}$, then $\displaystyle\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$ ($\mathscr{F}$ is closed under countable unions).

The pair $(\Omega, \mathscr{F})$ is then called a *measurable space*.

Remark 3.1. Note that property (iii) also tells us that $\mathscr{F}$ is closed under countable intersections. Indeed, if $A_1, A_2, \ldots \in \mathscr{F}$, then $A_1^c, A_2^c, \ldots \in \mathscr{F}$ by property (ii), and therefore $\displaystyle\bigcup_{i=1}^{\infty} A_i^c \in \mathscr{F}$. However, using DeMorgan's law, we have

$$\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c = \bigcap_{i=1}^{\infty} A_i\,.$$

P.H. Bezandry and T. Diagana, *Almost Periodic Stochastic Processes*,
DOI 10.1007/978-1-4419-9476-9_3,

Thus, again by property (ii), $\bigcap_{i=1}^{\infty} A_i \in \mathscr{F}$.

Here are some elementary σ-fields on the set Ω.
1. $\mathscr{F}_1 = \left\{\emptyset, \Omega\right\}$.
2. $\mathscr{F}_2 = \left\{\emptyset, \Omega, A, A^c\right\}$ for some $A \neq \emptyset$ and $A \neq \Omega$.
3. $\mathscr{F}_3 = \mathscr{P}(\Omega) = \left\{\text{all subsets of } \Omega \text{, including } \Omega \text{ itself}\right\}$.

Remark 3.2. In general, if Ω is uncountable, it is not an easy task to describe $\mathscr{F}_3$ because it is simply too big, as it contains all possible subsets of Ω. However, $\mathscr{F}_3$ can be chosen to contain any set of interest.

Now, let $\mathscr{U}$ be a collection of subsets of Ω and define

$$\sigma(\mathscr{U}) = \bigcap_{\mathscr{G} \supseteq \mathscr{U}} \mathscr{G},$$

where the $\mathscr{G}$'s are σ-fields on Ω.
Then $\sigma(\mathscr{U})$ is a unique σ-field, called the σ-field generated by $\mathscr{U}$. There is no σ-field smaller than $\sigma(\mathscr{U})$ that includes $\mathscr{U}$. The *Borel σ-field* is generated by the collection of open sets of a topological space. The elements of this σ-field are called *Borel sets*. For instance, the Borel σ-field on $\mathbb{R}$ is generated by the intervals in $\mathbb{R}$ and is denoted by $\mathscr{B}(\mathbb{R})$.

Definition 3.2. A probability measure on a measurable space $(\Omega, \mathscr{F})$ is a set function $\mathbf{P} : \mathscr{F} \to [0,1]$ with the properties
(i) $\mathbf{P}(\Omega) = 1$;
(ii) if $A_1, A_2, \ldots \in \mathscr{F}$ and $(A_i)_{i=1}^{\infty}$ is disjoint (i.e., $A_i \bigcap A_j = \emptyset$ if $i \neq j$), then

$$\mathbf{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mathbf{P}(A_i).$$

The triple $(\Omega, \mathscr{F}, \mathbf{P})$ is then called a *probability space*. The subsets A of Ω which are elements of $\mathscr{F}$ are called *$\mathscr{F}$-measurable sets*. In a probability context, these sets are called *events* and we interpret $\mathbf{P}(A)$ as "the probability that the event A occurs." Note that if $\mathbf{P}(A) = 1$, we say that "A occurs *almost surely* (a.s.)."

Example 3.1. Let $\Omega = [0,1]$, $\mathscr{F} = \mathscr{B}([0,1])$, the Borel σ-field on $[0,1]$, and $\mathbf{P} = \lambda$, the Lebesgue measure on $[0,1]$. In this case, the open intervals of the form (a,b), where $0 < a < b < 1$, could be taken as the generator sets, and $\lambda\big((a,b)\big) = b - a$. Hence, the triple $\Big([0,1], \mathscr{B}([0,1]), \lambda\Big)$ is a probability space.

Definition 3.3. A probability space $(\Omega, \mathscr{F}, \mathbf{P})$ is said to be *complete* if for every $A \subset B$ such that $B \in \mathscr{F}$ and $\mathbf{P}(B) = 0$, then $A \in \mathscr{F}$.

We assume throughout the book that all probability spaces are complete.

3.1.2 Sequence of Events

For a sequence of events $A_i \in \mathscr{F}$, $i = 1, 2, \dots$ on this space, define the limit superior

$$\begin{aligned}\limsup_{i\to\infty} A_i &= \bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty} A_i \\ &= \Big\{\omega :\ \omega \in A_i \ \text{ for infinitely many } i\,'s\Big\} \\ &= \Big\{\omega :\ \omega \in A_i \ \text{ for infinitely often}\Big\} \\ &= \Big\{\omega :\ \omega \in A_i,\ \text{i.o.}\Big\}\,.\end{aligned}$$

Similarly, we define the limit inferior

$$\begin{aligned}\liminf_{i\to\infty} A_i &= \bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty} A_i \\ &= \Big\{\omega :\ \omega \in A_i \ \text{ for all but finitely many } i\,'s\Big\}\,.\end{aligned}$$

It is not difficult to show that $\limsup\limits_{i\to\infty} A_i$ and $\liminf\limits_{i\to\infty} A_i$ belong to $\mathscr{F}$ and that $\liminf\limits_{i\to\infty} A_i \subset \limsup\limits_{i\to\infty} A_i$.

If $\liminf\limits_{i\to\infty} A_i = \limsup\limits_{i\to\infty} A_i$, the sequence (A_i) is said to be *convergent* with limit A, where $A = \lim\limits_{i\to\infty} A_i$.

The following lemma due to Borel–Cantelli is extremely useful for the derivations of many limit theorems of probability theory.

Lemma 3.1. *(i) If (A_i) is a sequence of arbitrary events and $\sum\limits_{i=1}^{\infty}\mathbf{P}(A_i) < \infty$, then*

$$\mathbf{P}\Big(\limsup_{i\to\infty} A_i\Big) = 0\,.$$

(ii) If (A_i) is a sequence of independent events satisfying $\sum\limits_{i=1}^{\infty}\mathbf{P}(A_i) = \infty$, then

$$\mathbf{P}\Big(\limsup_{i\to\infty} A_i\Big) = 1\,.$$

Proof. (i) Note first that $\limsup\limits_{i} A_i \subset \bigcup\limits_{i=j}^{\infty} A_i$. Now, using the monotonicity and subadditivity of $\mathbf{P}$, we have

$$\mathbf{P}\Big(\limsup_{i} A_i\Big) \le \mathbf{P}\Big(\bigcup_{i=j}^{\infty} A_i\Big) \le \sum_{i=j}^{\infty}\mathbf{P}(A_i),\quad j = 1, 2, \dots,$$

which implies that the extreme right term in these inequalities tends to zero as $j \to \infty$ since the series $\sum_{i=1}^{\infty} \mathbf{P}(A_i)$ is assumed to be convergent. Hence, $\mathbf{P}\big(\limsup_i A_i\big) = 0$.

(ii) Note that

$$\begin{aligned} 1 - \mathbf{P}\big(\limsup_i A_i\big) &= \mathbf{P}\big\{\big(\limsup_i A_i\big)^c\big\} \\ &= \mathbf{P}\big(\liminf_i A_i^c\big) \\ &= \lim_{j\to\infty} \mathbf{P}\Big\{\bigcap_{i=j}^{\infty} A_i^c\Big\}. \end{aligned}$$

Since the A_i's are independent, we have that for every i,

$$\begin{aligned} \mathbf{P}\Big(\bigcap_{i=j}^{\infty} A_i^c\Big) &= \lim_{n\to\infty} \mathbf{P}\Big(\bigcap_{i=j}^{n} A_i^c\Big) \\ &= \lim_{n\to\infty} \prod_{i=j}^{n} \mathbf{P}\big(A_i^c\big) \\ &= \lim_{n\to\infty} \prod_{i=j}^{n} (1 - \mathbf{P}\big(A_i\big) \\ &\le \lim_{n\to\infty} \prod_{i=j}^{n} \exp\big[-\mathbf{P}(A_i)\big] = \lim_{n\to\infty} \exp\Big[-\sum_{i=j}^{\infty} \mathbf{P}(A_i)\Big]. \end{aligned}$$

Now, using the fact that the series $\sum_{i=1}^{\infty} \mathbf{P}(A_i)$ is divergent, we can conclude that the right-hand side of the last inequality approaches zero. This completes the proof.

Definition 3.4. A function $Y : \Omega \to \mathbb{R}$ is called $\mathscr{F}$*-measurable* if

$$Y^{-1}(U) := \Big\{\omega \in \Omega : Y(\omega) \in U\Big\} \in \mathscr{F}$$

for all $U \in \mathscr{B}(\mathbb{R})$.

We are now prepared to give a precise definition of a random variable.

Definition 3.5. An $\mathbb{R}$-valued *random variable* X is an $\mathscr{F}$-measurable function $X : \Omega \to \mathbb{R}$. Every random variable X induces a probability measure μ_X on $\mathbb{R}$, defined by

$$\mu_X(B) = \mathbf{P}(X^{-1}(B)), \;\; \forall B \in \mathscr{B}(\mathbb{R}).$$

μ_X is called the *distribution* of X.

Definition 3.6. Suppose that X is a random variable with $\int_\Omega \big|X(\omega)\big|\, d\mathbf{P}(\omega) < \infty$. Then the *expectation* of X is the number

$$\mathbf{E}\big[X\big] := \int_\Omega X(\omega)\,d\mathbf{P}(\omega) = \int_\mathbb{R} x\,d\mu_X(x)\,.$$

Here are some standard inequalities which will frequently be used throughout this book.

Proposition 3.1. (i) The Markov inequality:
If $h : \mathbb{R} \to (0,\infty)$ is a strictly positive, even function that increases in $(0,\infty)$ and $\mathbf{E}[h(X)] < \infty$, then

$$\mathbf{P}\big(\big|X\big| > a\big) \le \frac{\mathbf{E}\big[h(X)\big]}{h(a)}, \quad a > 0.$$

(ii) The Chebyshev Inequality:

$$\mathbf{P}\big(\big|X - \mathbf{E}X\big| > a\big) \le \frac{\mathrm{Var}\big[\mathrm{X}\big]}{a^2}, \quad a > 0.$$

(iii) The Cauchy–Schwarz Inequality:

$$\mathbf{E}\big|XY\big| \le \big(\mathbf{E}[X^2]\big)^{1/2}\big(\mathbf{E}[Y^2]\big)^{1/2}.$$

(iv) The Hölder Inequality:
If $1 < p < \infty$ and q is given by $1/p + 1/q = 1$, $\mathbf{E}|X|^p < \infty$, and $\mathbf{E}|X|^q < \infty$, then

$$\mathbf{E}\Big|XY\Big| \le \Big[\mathbf{E}|X|^p\Big]^{1/p}\Big[\mathbf{E}|X|^q\Big]^{1/q}.$$

(v) The Jensen Inequality:
Let f be a convex function on $\mathbb{R}$. If $\mathbf{E}|X|$ and $\mathbf{E}|f(X)|$ are finite, then

$$f\big(\mathbf{E}[X]\big) \le \mathbf{E}\big[f(X)\big]\,.$$

Proof. (i) For any $a > 0$, we have

$$\mathbf{E}h(X) = \int_\Omega h(X)\,d\mathbf{P} \ge \int_{\{|X|\ge a\}} h(X)\,d\mathbf{P} \ge h(a)\mathbf{P}\big(|X| \ge a\big)\,,$$

which implies the desired result.

(ii) This property can be obtained from part (i) by replacing X by $X - \mathbf{E}[X]$ and taking $h(x) = x^2$.

(iii) & (iv) Property (iv) is an extension of the Hölder Inequality in the context of probability. Its proof is almost identical to that of Proposition 1.2 and may be omitted. As to property (iii), it is a particular case of (iv) with $p = q = 2$.

(v) To establish this property, let $l(x)$ be a tangent line to $f(x)$ at the point $x = \mathbf{E}[X]$. Write $l(x) = ax + b$ for some a and b. Now, by the convexity of f we have $f(x) \ge ax + b$. We then have

$$\begin{aligned}\mathbf{E}\big[f(\mathbf{E}[X])\big] &\geq \mathbf{E}[aX+b]\\ &= a\mathbf{E}[X]+b = l(\mathbf{E}[X])\\ &= f(\mathbf{E}[X])\,.\end{aligned}$$

The latter identity is true since l is tangent at the point to $f(x)$ at $\mathbf{E}[X]$. This completes the proof.

3.1.3 Convergence of Random Variables

Let X and X_n, $n = 1,2,\ldots$ be real-valued random variables defined on a probability space $(\Omega,\mathscr{F},\mathbf{P})$. The convergence of the sequence (X_n) toward X has various definitions depending on the way in which the difference between X_n and X is evaluated. In this subsection, we discuss the following modes of convergence: convergence in distribution, convergence in probability, almost sure convergence, and L^p convergence.

3.1.3.1 Convergence in Distribution

Definition 3.7. The sequence (X_n) converges in distribution to X if for all continuity points x of distribution F_X,

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)\,.$$

Here, F_{X_n}, F_X are the cumulative distribution functions of X_n and X, respectively.

Among the important characterizations of convergence in distribution is the following.

Proposition 3.2. *The sequence (X_n) converges in distribution to the random variable X if and only if for all bounded, continuous functions f,*

$$\lim_{n\to\infty} \mathbf{E}\big[f(X_n)\big] = \mathbf{E}\big[f(X)\big]\,.$$

It is well known that convergence in distribution is equivalent to pointwise convergence of the corresponding characteristic functions:

(X_n) converges in distribution to X if and only if $\lim_{n\to\infty} \mathbf{E}\big[e^{itX_n}\big] = \mathbf{E}\big[e^{itX}\big]$.

Also, note that although we talk of a sequence of random variables converging in distribution, it is really the distributions of those random variables that converge, not the random variables themselves.

3.1.3.2 Convergence in Probability

Definition 3.8. The sequence (X_n) *converges in probability* to the random variable X if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\Big\{\big|X_n - X\big| > \varepsilon\Big\} = 0\,.$$

Example 3.2. (Convergence in distribution, not in probability) Consider a sequence $(X_n)_{n\geq 0}$ of independent random variables defined on the probability space $(\Omega, \mathscr{F}, \mathbf{P})$ taking the values one and zero with probabilities $\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = 0) = \frac{1}{2}$. This sequence converges to X_0 in distribution but does not converge in probability to X_0.

To see this, let us compute the cumulative density function of X_n. We have

$$F_{X_n}(t) = \begin{cases} 0 \text{ if } t \leq 0, \\ \frac{1}{2} \text{ if } 0 < t \leq 1, \\ 1 \text{ if } t > 1. \end{cases}$$

Clearly, $F_{X_n}(t) = F_{X_0}(t)$ for all n and t. Therefore, (X_n) converges in distribution to X_0. However, for $n \neq 0$, note that

$$X_n - X_0 = \begin{cases} -1 \text{ with probability } \frac{1}{4}, \\ 0 \quad \text{with probability } \frac{1}{2}, \\ 1 \quad \text{with probability } \frac{1}{4}. \end{cases}$$

Let us now compute $\mathbf{P}\Big\{\big|X_n - X_0\big| > \frac{1}{2}\Big\}$. We then have

$$\begin{aligned} \mathbf{P}\Big\{\big|X_n - X_0\big| > \frac{1}{2}\Big\} &= \mathbf{P}\Big\{(X_n - X_0 < -\frac{1}{2}) \cup (X_n - X_0 > \frac{1}{2})\Big\} \\ &= \mathbf{P}\Big\{(X_n - X_0 = -1) \cup (X_n - X_0 = 1)\Big\} \\ &= \mathbf{P}\Big\{(X_n - X_0 = -1)\Big\} + \mathbf{P}\Big\{(X_n - X_0 = 1)\Big\} \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\,. \end{aligned}$$

Now, take $\varepsilon = \frac{1}{2}$. We obtain

$$\lim_{n\to\infty} \mathbf{P}\Big\{\big|X_n - X_0\big| > \frac{1}{2}\Big\} = \frac{1}{2} \neq 0\,.$$

Hence, (X_n) does not converge in probability to X_0.

The above example shows that the convergence in distribution does not imply the convergence in probability. However, we can establish the following.

Proposition 3.3. *If a sequence (X_n) converges in probability to X, then it converges in distribution to X.*

Proof. Let t be a point of continuity of F. Then,

$$\begin{aligned}\mathbf{P}\big(X \le t-\eta\big) &= \mathbf{P}\big(X \le t-\eta,\, X_n \le t\big) + \mathbf{P}\big(X \le t-\eta, X_n > t\big) \\ &\le \mathbf{P}\big(X_n \le t\big) + \mathbf{P}\big(\big|X_n - X\big| > \eta\big)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{P}\big(X_n \le t\big) &= \mathbf{P}\big(X_n \le t,\, X > t+\eta\big) + \mathbf{P}\big(X_n \le t, X \le t+\eta\big) \\ &\le \mathbf{P}\big(\big|X_n - X\big| > \eta\big) + \mathbf{P}\big(X \le t+\eta\big)\,.\end{aligned}$$

Since F is continuous at t, we have

$$\mathbf{P}\big(X \le t+\eta\big) \le F(t) + \varepsilon$$

and

$$\mathbf{P}\big(X \le t-\eta\big) \ge F(t) - \varepsilon\,.$$

Combining these inequalities, we obtain

$$F(t) - \varepsilon - \mathbf{P}\big(\big|X_n - X\big| > \varepsilon\big) \le \mathbf{P}\big(X_n \le t\big) \le F(t) + \varepsilon + \mathbf{P}\big(\big|X_n - X\big| > \varepsilon\big)\,.$$

Now, letting $n \to \infty$ and using the fact that $X_n \to X$ in probability, we obtain

$$F(t) - \varepsilon \le \lim_{n\to\infty} \mathbf{P}\big(X_n \le t\big) \le F(t) + \varepsilon\,,$$

which implies the desired result.

The following proposition shows that the converse of Proposition 3.3 is true if X is degenerate.

Proposition 3.4. *If (X_n) converges to the constant μ in distribution, then (X_n) converges to μ in probability.*

Proof. Suppose that (X_n) converges to μ in distribution. Then, we have

$$F_{X_n}(t) = \mathbf{P}(X_n \le t) \to 0 \ \text{ for all } t < \mu \tag{3.1}$$

and

$$F_{X_n}(t) = \mathbf{P}(X_n \le t) \to 1 \ \text{ for all } t > \mu\,. \tag{3.2}$$

It follows from (3.1) and (3.2) that for any $\varepsilon > 0$, one can find n_0 and n_1 such that

$$\mathbf{P}(X_n \le \mu - \varepsilon) < \frac{\varepsilon}{2} \ \text{ for any } n \ge n_0$$

and

$$\mathbf{P}(X_n > \mu + \varepsilon) < \frac{\varepsilon}{2} \text{ for any } n \geq n_1 \,.$$

Now, let $N = \max(n_0, n_1)$. Then, for any $n \geq N$,

$$\begin{aligned}
\mathbf{P}\big(|X_n - \mu| > \varepsilon\big) &= \mathbf{P}\big(X_n - \mu < -\varepsilon\big) + \mathbf{P}\big(X_n - \mu > \varepsilon\big) \\
&= \mathbf{P}\big(X_n < \mu - \varepsilon\big) + \mathbf{P}\big(X_n > \mu + \varepsilon\big) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \,.
\end{aligned}$$

Thus, (X_n) converges to μ in probability.

3.1.3.3 Almost Sure Convergence

Definition 3.9. The sequence (X_n) *converges almost surely* (a.s.) to the random variable X if

$$\mathbf{P}\Big\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\Big\} = 1\,.$$

Example 3.3. Let sample space Ω be the closed unit interval $[0,1]$ with the uniform probability distribution λ. Define random variables $X_n(\omega) = \omega + \omega^n$ and $X(\omega) = \omega$. For every $\omega \in [0,1)$, $\omega^n \to 0$ as $n \to \infty$ and $X_n(\omega) \to \omega$. However, since $X_n(1) = 2$ for every n, $X_n(1)$ does not converge to $1 = X(1)$. But, since the convergence occurs on the set $[0,1)$ and $\lambda(\{1\}) = 0$, (X_n) converges to X almost surely.

Note that the almost sure convergence has some equivalent definitions. For instance, it can be easily shown that $X_n \to X$ a.s. if and only if for any $\varepsilon > 0$ we have

$$\lim_{m\to\infty} \mathbf{P}\big\{|X_n - X| > \varepsilon \text{ for some } n \geq m\big\} = 0\,. \tag{3.3}$$

The following proposition provides an important sufficient condition for almost sure convergence.

Proposition 3.5. *If* $\sum_{n=1}^{\infty} \mathbf{P}\big(|X_n - X| > \varepsilon\big) < \infty$ *for every* $\varepsilon > 0$, *then the sequence* (X_n) *converges to* X *almost surely.*

Proof. Let $E_n(\varepsilon) = \big\{|X_n - X| \geq \varepsilon\big\}$. Then, by assumption, the series $\sum_{n=1}^{\infty} \mathbf{P}(E_n(\varepsilon))$ is convergent. By Proposition 3.1(i), $\mathbf{P}\big(\limsup_n E_n(\varepsilon)\big) = 0$ for each $\varepsilon > 0$, which implies the desired result.

Using property (3.3), we can now establish the following.

Proposition 3.6. *If a sequence* (X_n) *converges almost surely to* X, *then it converges in probability to* X.

Proof. Let $E_i(k) = \big\{|X_i - X| \geq \frac{1}{k}\big\}$. Then $A_{nk} = \bigcup_{i=n}^{\infty} E_i(k)$ is the set of those ω such that $|X_i(\omega) - X(\omega)| \geq \frac{1}{k}$ for all $i \geq n$, and observe that

$$\mathbf{P}\big(A_{nk}\big) \geq \mathbf{P}\big(E_n(k)\big) \text{ for all } n,\, k \geq 1\,.$$

Now, since $X_n \to X$ a.s., $\mathbf{P}\big(A_{nk}\big) \to 0$ for all k. The latter property shows that $\mathbf{P}\big(E_n(k)\big) \to 0$ for all k, which implies the desired result.

The converse of Proposition 3.6 is false. However, we can establish the following proposition.

Proposition 3.7. *If (X_n) converges in probability to X, there exists a suitable subsequence (n_k) such that (X_{n_k}) converges almost surely to X.*

Proof. Pick an increasing sequence (n_k) such that

$$\mathbf{P}\big\{|X_{n_k} - X| > \frac{1}{k}\big\} \leq \frac{1}{k^2}\,.$$

This can be done since $X_n \to X$ in probability. Then, for any $\varepsilon > 0$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbf{P}\big(|X_{n_k} - X| > \varepsilon\big) &\leq \sum_{k:\,k<\varepsilon^{-1}} \mathbf{P}\big(|X_{n_k} - X| > \varepsilon\big) + \sum_{k:\,k\geq\varepsilon^{-1}} \mathbf{P}\big(|X_{n_k} - X| > \frac{1}{k}\big) \\ &\leq \sum_{k:\,k<\varepsilon^{-1}} \mathbf{P}\big(|X_{n_k} - X| > \varepsilon\big) + \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty\,.\end{aligned}$$

Consequently, $X_{n_k} \to X$ a.s. as $k \to \infty$ by Proposition 3.5.

The following proposition extends some properties of algebraic operations on convergent sequences of real numbers to sequences of random variables.

Theorem 3.1. (Slutsky's Theorem) *If X_n converges in distribution to X and Y_n converges to a, a constant, in probability, then*
(a) $X_n + Y_n$ converges $X + a$ in distribution.
(b) $Y_n X_n$ converges to aX in distribution.

Proof. (a) We may assume that $a = 0$. Let x be a continuity point of the cumulative distribution function F_X of X. We then have

$$\begin{aligned}\mathbf{P}\big(X_n + Y_n \leq x\big) &\leq \mathbf{P}\big(X_n + Y_n \leq x, |Y_n| \leq \varepsilon\big) + \mathbf{P}\big(X_n + Y_n \leq x, |Y_n| > \varepsilon\big) \\ &\leq \mathbf{P}\big(X_n \leq x + \varepsilon\big) + \mathbf{P}\big(|Y_n| > \varepsilon\big)\,.\end{aligned}$$

Similarly,

$$\mathbf{P}\big(X_n \leq x - \varepsilon\big) \leq \mathbf{P}\big(X_n + Y_n \leq x\big) + \mathbf{P}\big(|Y_n| > \varepsilon\big)\,.$$

Hence,

$$\mathbf{P}\big(X_n \leq x - \varepsilon\big) - \mathbf{P}\big(|Y_n| > \varepsilon\big) \leq \mathbf{P}\big(X_n + Y_n \leq x\big) \leq \mathbf{P}\big(X_n \leq x + \varepsilon\big) + \mathbf{P}\big(|Y_n| > \varepsilon\big)\,.$$

Letting $n \to \infty$ and then $\varepsilon \to 0$ proves (a).

(b) To prove this property, we use Proposition 3.2. We prove that

$$\mathbf{E}\big[f(X_nY_n)\big] \to \mathbf{E}\big[f(aX)\big]$$

for every bounded continuous function f.

Let $M = \sup_x |f(x)| < \infty$, fix $\varepsilon > 0$, and choose K such that $\pm K$ are continuity points of the cumulative distribution function F_X and $\mathbf{P}\big(|X| > K\big) < \frac{\varepsilon}{16}$, which implies that $\mathbf{P}\big(|X_n| > K\big) < \frac{\varepsilon}{8}$ for all sufficiently large values of n. Then, one can find $\eta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{4}$ whenever $|x-y| < \eta$. Also, the convergence in probability of the sequence (Y_n) toward a constant a allows us to choose $N_0 > 0$ such that

$$\mathbf{P}\big\{|Y_n - a| > \frac{\eta}{K}\big\} < \frac{\varepsilon}{8M}$$

whenever $n \geq N_0$. We then have

$$\begin{aligned}
\Big|\mathbf{E}\big[f(X_nY_n)\big] - \mathbf{E}\big[f(aX)\big]\Big| &\leq \mathbf{E}\Big[\big|f(X_nY_n) - f(aX_n)\big|;\ |Y_n - a| > \frac{\eta}{K}\Big] \\
&\quad + \mathbf{E}\Big[\big|f(X_nY_n) - f(aX_n)\big|; \big\{|Y_n - a| \leq \frac{\eta}{K},\ |X_n| > K\big\}\Big] \\
&\quad + \mathbf{E}\Big[\big|f(X_nY_n) - f(aX_n)\big|; \big\{|Y_n - a| \leq \frac{\eta}{K},\ |X_n| \leq K\big\}\Big] \\
&\quad + \Big|\mathbf{E}\big[f(aX_n)\big] - \mathbf{E}\big[f(aX)\big]\Big| \\
&\leq 2M\,\mathbf{P}\big\{|Y_n - a| > \frac{\eta}{K}\big\} + 2M\mathbf{P}\big(|X_n| > K\big) + \frac{\varepsilon}{4} \\
&\quad + \Big|\mathbf{E}\big[f(aX_n)\big] - \mathbf{E}\big[f(aX)\big]\Big|.
\end{aligned}$$

On the other hand, we can show that the sequence (aX_n) converges to aX in distribution. Indeed, take a bounded and continuous function $h(x) = f(ax)$ and use the fact that the sequence (X_n) converges X in distribution. It follows that

$$\mathbf{E}\big[f(aX_n)\big] = \mathbf{E}\big[h(X_n)\big] \to \mathbf{E}\big[h(X)\big] = \mathbf{E}\big[f(aX)\big].$$

The latter allows us to choose $N_1 > 0$ such that

$$\Big|\mathbf{E}\big[f(aX_n)\big] - \mathbf{E}\big[f(aX)\big]\Big| < \frac{\varepsilon}{4}$$

whenever $n \geq N_1$.

Now, take $N = \max(N_0, N_1)$. For any $n \geq N$, we obtain

$$\Big|\mathbf{E}\big[f(X_nY_n)\big] - \mathbf{E}\big[f(aX)\big]\Big| \leq \frac{2\varepsilon}{8} + \frac{2\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,$$

as desired.

The following proposition due to Skorohod relates convergence in distribution and almost sure convergence.

Proposition 3.8. (Skorohod Representation Theorem) *Let (X_n) be a sequence of random variables, and assume that (X_n) converges to X in distribution as $n \to \infty$. Let F_n be the cumulative distribution function of X_n and let F be the cumulative distribution function of X. Then, there exists a probability space $(\Omega', \mathscr{F}', \mathbf{P}')$ and random variables Y_n and Y all defined on $(\Omega', \mathscr{F}', \mathbf{P}')$ such that Y has cumulative distribution F and each F_n has cumulative distribution function F_n, and (Y_n) converges to Y almost surely as $n \to \infty$.*

Proof. For a proof, see. e.g., Billingsley [29].

3.1.3.4 L^p-Convergence

Definition 3.10. Let $p \geq 1$. The sequence (X_n) *converges in L^p* to the random variable X if $\mathbf{E}\big|X_n\big|^p + \mathbf{E}\big|X\big|^p < \infty$ for all n and

$$\lim_{n\to\infty} \mathbf{E}\big|X_n - X\big|^p = 0\,.$$

By Markov's inequality, $\mathbf{P}\Big\{\omega : \big|X_n(\omega) - X(\omega)\big| > \varepsilon\Big\} \leq \dfrac{1}{\varepsilon^p}\mathbf{E}\big|X_n - X\big|^p$ for any $\varepsilon > 0$. Thus, if (X_n) converges in L^p to X, then (X_n) converges in probability to X. The converse is in general false.

Example 3.4. (Convergence in probability, not in L^p)

Let $([0,1], \mathscr{B}([0,1]), \mathbf{P})$ be a probability space with $\mathbf{P}(d\omega) = d\omega$, the uniform probability distribution on $[0,1]$, and let $X_n = 2^n\,\mathbf{1}_{(0,\frac{1}{n})}$ be a sequence of random variables defined on this space. The sequence X_n converges in probability to zero as $n \to \infty$ but does not converge in L^p, $p \geq 1$.

To see this, fix $\varepsilon > 0$. Then we have

$$\mathbf{P}\big(|X_n| > \varepsilon\big) = \mathbf{P}(0, \frac{1}{n}) = \frac{1}{n} \to 0 \;\text{ a.s. } n \to \infty\,.$$

Hence, (X_n) converges in probability to zero. On the other hand,

$$\mathbf{E}|X_n|^p = 2^{np}\,\mathbf{P}(0, \frac{1}{n}) = \frac{2^{np}}{n} \to \infty \;\text{ a.s. } n \to \infty\,.$$

Hence, convergence in probability does not imply L^p-convergence.

Example 3.5. (Convergence in L^p, not almost surely) Let sample space Ω be the closed unit interval $[0,1]$ with Lebesgue measure λ. Define a sequence (X_n) of random variables as follows: $X_1 = \mathbf{1}_{[0,\frac{1}{2}]}$, $X_2 = \mathbf{1}_{[\frac{1}{2},1]}$, $X_3 = \mathbf{1}_{[0,\frac{1}{3}]}$, $X_4 = \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]}$,

$X_5 = \mathbf{1}_{[\frac{2}{3},1]}$, and so on. We claim that this sequence converges to zero in L^p but does not converge to zero almost surely.

To see this, let us compute the pth moment of some random variables and observe the pattern. We have

$$\mathbf{E}\big|X_1\big|^p = \mathbf{E}\big|X_2\big|^p = \frac{1}{2}, \ \ \mathbf{E}\big|X_3\big|^p = \mathbf{E}\big|X_4\big|^p = \mathbf{E}\big|X_5\big|^p = \frac{1}{3}, \ldots$$

so that $\mathbf{E}\big|X_n\big|^p \to 0$ as $n \to \infty$. In addition, by Chebyshev's inequality, the latter implies that (X_n) converges in probability to 0. However, (X_n) does not converge to 0 almost surely. Indeed, there is no value of $\omega \in [0,1]$ for which $X_n(\omega) \to 0$. For every $\omega \in [0,1]$, the value of $X_n(\omega)$ alternates between 0 and 1 infinitely often. No pointwise convergence occurs for this sequence.

Remark 3.3. For proofs of the various results discussed in this subsection, we refer the reader to for instance Bauer [17], Casella and Berger [34], or Métivier [140].

3.1.4 Conditional Expectation

Let X be an integrable random variable defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and $\mathscr{G}$ denote a sub-σ-field of $\mathscr{F}$.

Definition 3.11. The *conditional expectation* $\mathbf{E}\big[X \,|\, \mathscr{G}\big]$ of X with respect to $\mathscr{G}$ is defined to be the class of $\mathscr{G}$-measurable functions satisfying

$$\int_A X \, d\mathbf{P} = \int_A \mathbf{E}\big[X \,|\, \mathscr{G}\big] \, d\mathbf{P}, \ \ \forall A \in \mathscr{G}. \tag{3.4}$$

It is important to note that the random variable $\mathbf{E}\big[X \,|\, \mathscr{G}\big]$ can be understood as an updated version of the expectation of X, given the information $\mathscr{F}$.

We list here some properties of the conditional expectation $\mathbf{E}\big[X \,|\, \mathscr{G}\big]$ that are frequently used in calculations.

Proposition 3.9. *(i)* Expectation Law*:* $\mathbf{E}\Big[\mathbf{E}\big[X \,|\, \mathscr{G}\big]\Big] = \mathbf{E}\Big[X\Big]$.

(ii) If X is $\mathscr{G}$-measurable, then $\mathbf{E}\big[X \,|\, \mathscr{G}\big] = X$ a.s.

(iii) Stability*: If Y is $\mathscr{G}$-measurable and bounded, then*

$$\mathbf{E}\big[XY \,|\, \mathscr{G}\big] = Y\mathbf{E}\big[X \,|\, \mathscr{G}\big] \ \ a.s. \ \ \forall Y \in \mathscr{G}.$$

(iv) Independence Law*: If X and the σ-field $\mathscr{G}$ are independent, then* $\mathbf{E}\big[X \,|\, \mathscr{G}\big] = \mathbf{E}[X]$.

(v) $\mathbf{E}\Big[\big(X - \mathbf{E}\big[X \,|\, \mathscr{G}\big]\big)Y\Big] = 0, \ \ \forall Y \in \mathscr{G}.$

(vi) The conditional expectation $\mathbf{E}\big[X \,|\, \mathscr{G}\big]$ is the projection of X on $\mathscr{G}$ and $X - \mathbf{E}\big[X \,|\, \mathscr{G}\big]$ is orthogonal to $\mathscr{G}$. In other words, $\mathbf{E}\big[X \,|\, \mathscr{G}\big]$ is the $\mathscr{G}$-measurable random variable that is closest to X in the mean square sense.

Proof. (i) This property follows immediately from (3.4) with $A = \Omega$.

(ii) This property follows from the fact that X is $\mathscr{G}$-measurable and $\mathbf{E}[X1_A] = \mathbf{E}[X1_A]$ for all $A \in \mathscr{G}$.

(iii) To prove this property, we show that $Y\mathbf{E}\big[X|\mathscr{G}\big]$ is a version of $\mathbf{E}\big[XY|\mathscr{G}\big]$. The $\mathscr{G}$-measurability of $Y\mathbf{E}\big[X|\mathscr{G}\big]$ follows from that of Y and $\mathbf{E}\big[X|\mathscr{G}\big]$. It remains to show that

$$\int_A Y\mathbf{E}\big[X|\mathscr{G}\big]\,d\mathbf{P} = \int_A YX\,d\mathbf{P},\ \ A \in \mathscr{G}\,. \tag{3.5}$$

If $Y = 1_G$, $G \in \mathscr{G}$, then for any $H \in \mathscr{G}$, one has

$$\int_H \mathbf{E}\big[X1_G|\mathscr{G}\big]\,d\mathbf{P} = \int_H X1_G\,d\mathbf{P} = \int_{H\cap G} X\,d\mathbf{P}$$

and

$$\int_H 1_G\mathbf{E}\big[X|\mathscr{G}\big]\,d\mathbf{P} = \int_{G\cap H} \mathbf{E}\big[X|\mathscr{G}\big]\,d\mathbf{P} = \int_{G\cap H} X\,d\mathbf{P}\,.$$

Hence, (3.5) holds for this case. One can also show that (3.5) holds for a simple random variable $Y = \sum_n \alpha_n 1_{A_n}$, $A_n \in \mathscr{G}$ by linearity of expectation. The extension to any random variable follows immediately from the representation of Y by a difference of two positive random variables, which can be defined as limits of simple random variables.

(iv) Let $A \in \mathscr{G}$. Then by independence

$$\int_A X\,d\mathbf{P} = \int 1_A X\,d\mathbf{P} = \mathbf{E}\big[1_A X\big] = \mathbf{E}[1_A]\mathbf{E}[X] = \int_A \mathbf{E}[X]\,d\mathbf{P}$$

from which the property follows.

(v) The proof of this property is similar to that of property (iii). It is left to the reader as an exercise.

(vi) This property is a straight consequence of property (v).

We now collect essential properties of the conditional expectation that are similar to the properties of the expectation operator.

Proposition 3.10. *If X and X_n are integrable random variables, then*

*(i)*Linearity*:* $\mathbf{E}\Big[aX_1 + bX_2\,|\,\mathscr{G}\Big] = a\mathbf{E}\big[X_1\,|\,\mathscr{G}\big] + b\mathbf{E}\big[X_2\,|\,\mathscr{G}\big]$ a.s.

(ii) Positivity*: $X \geq 0$ implies* $\mathbf{E}\big[X\,|\,\mathscr{G}\big] \geq 0$ a.s.

(iii) Monotonicity*: $X_1 \leq X_2$ implies* $\mathbf{E}\big[X_1\,|\,\mathscr{G}\big] \leq \mathbf{E}\big[X_2\,|\,\mathscr{G}\big]$ a.s.

(iv) Monotone convergence*: if $X_n \uparrow X$ a.s., then* $\mathbf{E}\big[X_n\,|\,\mathscr{G}\big] \uparrow \mathbf{E}\big[X\,|\,\mathscr{G}\big]$ a.s.

*(v)*Dominated convergence*: $|X_n| \leq Y,\ \mathbf{E}[Y] < \infty$, and $X_n \to X$ a.s. imply* $\mathbf{E}\big[X_n\,|\,\mathscr{G}\big] \to \mathbf{E}\big[X\,|\,\mathscr{G}\big]$.

(vi) Cauchy–Schwarz inequality*:* $\Big(\mathbf{E}\big[XY\,|\,\mathscr{G}\big]\Big)^2 \leq \mathbf{E}\big[X^2\,|\,\mathscr{G}\big]\,\mathbf{E}\big[Y^2\,|\,\mathscr{G}\big]$a.s.

(vii) Jensen inequality*:* $\psi\big(\mathbf{E}\big[X\,|\,\mathscr{G}\big]\big) \leq \mathbf{E}\big[\psi(X)\,|\,\mathscr{G}\big]$ a.s. *for a convex function ψ.*

(viii) Modulus inequality*:* $\big|\mathbf{E}\big[X\,|\,\mathscr{G}\big]\big| \leq \mathbf{E}\big[|X|\,|\,\mathscr{G}\big]$ a.s.

Proof. (i) This property follows immediately from the linearity of the integral.

(ii) To prove this, let $X \geq 0$. Then, for every $A \in \mathscr{G}$, we have

$$\int_A \mathbf{E}\big[X|\mathscr{G}\big]\, d\mathbf{P} = \int_A X\, d\mathbf{P} \geq 0$$

so that $\mathbf{E}\big[X|\mathscr{G}\big] \geq 0$ a.s.

(iii) This property follows immediately from (ii).

(iv) By monotonicity, there is a $\mathscr{G}$-measurable random variable Y such that $\mathbf{E}\big[X_n|\mathscr{G}\big] \uparrow Y$. Let $A \in \mathscr{G}$. Using the Lebesgue Monotone Convergence Theorem, one has

$$\mathbf{E}\big[Y 1_A\big] = \lim_{n\to\infty} \mathbf{E}\big[\mathbf{E}\big[X_n|\mathscr{G}\big]\, 1_A\big] = \lim_{n\to\infty} \mathbf{E}\big[X_n 1_A\big] = \mathbf{E}\big[X 1_A\big]\,,$$

which proves that $Y = \mathbf{E}\big[X|\mathscr{G}\big]$.

(v) To prove this property, let $Y_n = \sup_{k\geq n} |X_k - X|$, $n \geq 1$. Then $Y_n \geq 0$, $Y_n \downarrow$ as $n \to \infty$ and $Y_n \leq 2Z$ almost surely for $n \geq 1$ so that Y_n is integrable for each $n \geq 1$. Also, since X_n converges to X almost surely, Y_n converges to 0 almost surely. On the other hand, we have

$$\big|\mathbf{E}\big(X_n|\mathscr{G}\big) - \mathbf{E}\big(X|\mathscr{G}\big)\big| \leq \mathbf{E}\big[|X_n - X||\mathscr{G}\big] \leq \mathbf{E}[Y_n] \text{ a.s.}$$

Thus, it is sufficient to show that $\lim_{n\to\infty} \mathbf{E}\big[Y_n|\mathscr{G}\big] = 0$ a.s. From the fact that $Y_n \geq 0$ and $Y_n \downarrow$ it follows that $\mathbf{E}\big[Y_n|\mathscr{G}\big] \geq 0$ and $\mathbf{E}\big[Y_n|\mathscr{G}\big] \downarrow$ and hence $V = \lim_{n\to\infty} \mathbf{E}\big[Y_n|\mathscr{G}\big]$ exists and $\mathbf{E}[V] \leq \mathbf{E}\big[\mathbf{E}\big[Y_n|\mathscr{G}\big]\big] = \mathbf{E}[Y_n]$. But $\lim_{n\to\infty} Y_n = 0$ a.s. and $Y_n \leq 2Z$ so that by the Dominated Convergence Theorem $\lim_{n\to\infty} \mathbf{E}[Y_n] = 0$.

(vi) Define the random variables

$$U = \big(\mathbf{E}\big[|X|^2|\mathscr{G}\big]\big)^{1/2},\ V = \big(\mathbf{E}\big[|Y|^2|\mathscr{G}\big]\big)^{1/2}$$

and note that they are $\mathscr{G}$-measurable. Observe

$$\mathbf{E}\big[|X|^2 1_{U=0}\big] = \mathbf{E}\big[1_{U=0}\mathbf{E}\big[|X|^2\big]|\mathscr{G}\big] = \mathbf{E}\big[1_{U=0}\, U^2\big] = 0\,.$$

Thus, $|X| 1_{U=0} = 0$ a.s., which implies that

$$\mathbf{E}\big[|XY||\mathscr{G}\big] 1_{U=0} = \mathbf{E}\big[|XY|\, 1_{U=0}|\mathscr{G}\big] = 0\,.$$

Similarly, we can also show that $\mathbf{E}\big[|XY||\mathscr{G}\big]\, 1_{V=0} = 0$. Therefore, the conditional Cauchy–Schwarz holds on the set $\{U = 0\} \cup \{V = 0\}$.

On the set $\big\{U = \infty, V > 0\big\} \cup \big\{U > 0, V = \infty\big\}$, the right-hand side is infinite and the conditional Cauchy–Schwarz inequality holds too. Dividing by the right-hand side, it is then enough to show that $\dfrac{\mathbf{E}\big[|XY||\mathscr{G}\big]}{UV}\, 1_H \leq 1$ a.s. on the set $H := \big\{0 < U < \infty, 0 < V < \infty\big\}$.

To prove this, let $G \in \mathscr{G}$, $G \subset H$. Using the measurability of U, V, and 1_G with respect to $\mathscr{G}$, the properties of the conditional expectation, and classical Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\mathbf{E}\Big[\frac{\mathbf{E}\big[|XY|\,|\mathscr{G}\big]}{UV}\,1_G\Big] &= \mathbf{E}\Big[\mathbf{E}\Big[\frac{|XY|}{UV}\,1_G|\mathscr{G}\Big]\Big] \\
&= \mathbf{E}\Big[\frac{|X|}{U}\,1_G \cdot \frac{|Y|}{V}\,1_G\Big] \\
&\le \Big(\mathbf{E}\Big[\frac{|X|^2}{U^2}\,1_G\Big]\Big)^{1/2}\Big(\mathbf{E}\Big[\frac{|Y|^2}{V^2}\,1_G\Big]\Big)^{1/2} \\
&\le \Big(\mathbf{E}\Big[\frac{\mathbf{E}\big[|X|^2|\mathscr{G}\big]}{U^2}\,1_G\Big]\Big)^{1/2}\Big(\mathbf{E}\Big[\frac{\mathbf{E}\big[|Y|^2|\mathscr{G}\big]}{V^2}\,1_G\Big]\Big)^{1/2} \\
&= \big(\mathbf{E}[1_G]\big)^{1/2}\big(\mathbf{E}[1_G]\big)^{1/2} = \mathbf{E}[1_G]\,,
\end{aligned}$$

which implies the desired result.

(vii) To prove this property, we use a classical characterization of a convex function, namely, every convex function Ψ is the upper envelope of a countable collection of such lines: $a_n x + b_n$, $n \ge 1$. Define $L_n(x) = a_n x + b_n$, for all x. We then have

$$L_n\big(X|\mathscr{G}\big) = \mathbf{E}\big[L_n(X)|\mathscr{G}\big] \le \mathbf{E}\big[\Psi(X)|\mathscr{G}\big]$$

and thus

$$\Psi\big(\mathbf{E}\big[X|\mathscr{G}\big]\big) = \sup_n L_n\big(\mathbf{E}\big[X|\mathscr{G}\big]\big) \le \mathbf{E}\big[\Psi(X)|\mathscr{G}\big]\,.$$

(viii) The proof of this property is left to the reader as an exercise.

3.2 Stochastic Processes

In recent years there has been an ever-increasing interest in the study of systems which evolve in time in a random manner. Mathematical models of such systems are known as stochastic processes.

More precisely, let X be the random variable of interest depending on a parameter t, which assumes values from a set $\mathbb{T} \subset [0,\infty)$. In many applications, the parameter t is considered to be time. Thus, $X(t)$ is the state of random variable X at time t, whereas $\mathbb{T}$ denotes the time set. Furthermore, let $\mathbb{S}$ denote the set of all states (realizations) which the $X(t)$, $t \in \mathbb{T}$, can assume.

Definition 3.12. A *stochastic process* X with parameter set $\mathbb{T}$ and state space $\mathbb{R}$ is a collection of $\mathbb{R}$-valued random variables

$$\Big\{X(t),\, t \in \mathbb{T}\Big\} = \Big\{X(\omega,t),\ \omega \in \Omega,\ t \in \mathbb{T}\Big\}$$

defined on the probability space $(\Omega, \mathscr{F}, \mathbf{P})$.

Note that for each fixed $t \in \mathbb{T}$ we have a random variable

$$\omega \to X(\omega,t)\,.$$

On the other hand, for each fixed $\omega \in \Omega$ we have a function

$$t \to X(\omega,t)\,,$$

which is called a *sample path* of the process. The stochastic process X may be regarded as a function of two variables (ω,t) from $\Omega \times \mathbb{T}$ to $\mathbb{R}$.

If $\mathbb{T}$ is a finite or countably infinite set, then $\{X(t),\, t \in \mathbb{T}\}$ is called a *discrete-time* stochastic process. Such processes can be written as a sequence of random variables (X_t). Conversely, every sequence of random variables can be interpreted as a discrete-time stochastic process. If $\mathbb{T}$ is an interval, then $\{X(t),\, t \in \mathbb{T}\}$ is a *continuous-time* stochastic process. The stochastic process $\{X(t),\, t \in \mathbb{T}\}$ is said to be *discrete* if its state space $\mathbb{S}$ is a finite or countably infinite set. It is said to be *continuous* if $\mathbb{S}$ is an interval.

This section introduces three types of stochastic processes, Brownian motion, Gaussian processes, and martingales, that play a central role in the theory of stochastic processes.

3.2.1 Continuity

In this subsection, we give some of the most common definitions of continuity for stochastic processes.

Let $\{X(t),\, t \in \mathbb{T}\}$ be an $\mathbb{R}$-valued stochastic process on a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$.

Definition 3.13. (i) X is continuous in probability at $t \in \mathbb{T}$ if for any $\varepsilon > 0$,

$$\lim_{s \to t} \mathbf{P}\Big\{ \big|X(\omega,s) - X(\omega,t)\big| > \varepsilon \Big\} = 0.$$

(ii) X is continuous in the p-th mean at $t \in \mathbb{T}$ if

$$\lim_{s \to t} \mathbf{E}\Big[\big|X(s) - X(t)\big|^p \Big] = 0\,. \tag{3.6}$$

(iii) X is almost sure (a.s.) continuous at $t \in \mathbb{T}$ if

$$\mathbf{P}\Big\{ \omega \in \Omega : \lim_{s \to t} \big|X(\omega,s) - X(\omega,t)\big| = 0 \Big\} = 1\,. \tag{3.7}$$

Remark 3.4. (i) In Definition 3.13(iii), Eq. (3.7) is equivalent to

$$\mathbf{P}\Big\{ \omega \in \Omega : \lim_{s \to t} X(\omega,s) \neq X(\omega,t) \Big\} = 0\,.$$

(ii) If $p = 2$ in Eq. (3.6), X is said to be *continuous in the mean-square sense* at t. The p-th mean continuity is used extensively later in the following chapters.

The stochastic process X is continuous in probability, continuous in the p-th mean, and almost surely continuous in an interval $I \subset \mathbb{T}$ if it is continuous in probability, continuous in the p-th mean, and almost surely continuous at each $t \in I$, respectively.

Definition 3.14. Two stochastic processes X and Y with a common index $\mathbb{T} \subset \mathbb{R}_+$ are called *versions* of one another if for all $t \in \mathbb{T}$,

$$\mathbf{P}\Big\{\omega : X(\omega,t) = Y(\omega,t)\Big\} = 1\,.$$

Such processes are also said to be *stochastically equivalent*.

Proposition 3.11. *If X and Y are versions of one another, they have the same finite-dimensional distributions.*

Proof. Let $\mathbb{I}$ be an arbitrary finite collection of indices. It suffices to show that $\mathbf{P}\Big\{\omega : X_{\mathbb{I}}(\omega) = Y_{\mathbb{I}}(\omega)\Big\} = 1$. For this purpose let $\mathbb{I} = \big\{t_j,\, 1 \le j \le i\big\}$. Using additivity of $\mathbf{P}$, we have

$$\begin{aligned}
\mathbf{P}\Big\{\omega : X_{\mathbb{I}}(\omega) = Y_{\mathbb{I}}(\omega)\Big\} &= \mathbf{P}\Big\{\omega : X(\omega,t_1) = Y(\omega,t_1),\ldots,X(\omega,t_i) = Y(\omega,t_i)\Big\} \\
&= 1 - \mathbf{P}\Big\{\bigcup_{j=1}^{i}\Big(\omega : X(\omega,t_j) \neq Y(\omega,t_j)\Big)\Big\} \\
&\geq 1 - \sum_{j=1}^{i}\mathbf{P}\Big\{X(\omega,t_j) \neq Y(\omega,t_j)\Big\} \\
&= 1\,.
\end{aligned}$$

There is a stronger notion of similarity between processes than that of versions, which is sometimes useful in applications.

Definition 3.15. Two stochastic processes X and Y are *indistinguishable* if their sample paths coincide almost surely, that is,

$$\mathbf{P}\Big\{\omega : \forall t \in \mathbb{T},\, X(\omega,t) = Y(\omega,t)\Big\} = 1\,.$$

In the following example, we describe stochastic processes that are versions of one another, but not indistinguishable.

Example 3.6. Let $X = \big\{X(t),\, 0 \le t \le 1\big\}$ and $Y = \big\{Y(t),\, 0 \le t \le 1\big\}$ be real-valued stochastic processes defined on the probability space $\Big([0,1],\, \mathscr{B}([0,1]),\, \lambda\Big)$, where λ is the Lebesgue measure on $[0,1]$, such that $X(\omega,t) = 0$ and

$$Y(\omega,t) = \begin{cases} 1 \text{ if } \ \omega = t, \\ \\ 0 \text{ if } \ \omega \neq t. \end{cases}$$

Note that for each $\omega \in [0,1]$ fixed,

$$\sup_{0\leq t\leq 1} X(\omega,t) = 0 \ \text{ while } \ \sup_{0\leq t\leq 1} Y(\omega,t) = 1\,.$$

It follows that the sample paths of X and Y differ for $\omega \in [0,1]$. Therefore, they are not indistinguishable.
On the other hand, for each $t \in [0,1]$ fixed, let $\Omega_t = \{\omega : X(\omega,t) \neq Y(\omega,t)\} = \{t\}$. Then, we have $\lambda\left(\{t\}\right) = 0$, which means that the processes X and Y are versions of one another.

We now state a famous theorem of Kolmogorov.

Theorem 3.2. *Suppose that the process $X = \{X(t),\, t \in \mathbb{T}\}$ satisfies the following condition: for all $T > 0$ there exist positive constants α, β, and C such that*

$$\mathbf{E}\big|X(t) - X(s)\big|^{\alpha} \leq C\big|t-s\big|^{1+\beta}$$

for $0 \leq s, t \leq T$.
Then there exists a continuous version of X.

For a proof, see, e.g., Strook and Varadhan [168] or Bakstein and Capasso [15].

3.2.2 Separability and Measurability

Let $X = \{X(t),\, 0 \leq t \leq 1\}$ be a stochastic process. In general, $\sup_{0\leq t\leq 1} X(t)$ does not define a random variable. For instance, take $\Omega = [0,1]$, $\mathscr{F} = \mathscr{B}([0,1])$, and $\mathbf{P} = \lambda$, the Lebesgue measure on $[0,1]$. Let $A \subset [0,1]$ be a nonmeasurable set and define a stochastic process by

$$X(\omega,t) = \begin{cases} 1 \text{ if } t \in A \text{ and } \omega = t, \\ \\ 0 \text{ if } \ \text{otherwise}. \end{cases}$$

Then the function $\omega \to \sup_{0\leq t\leq 1} X(\omega,t)$ is given by

$$\sup_{0\leq t\leq 1} X(\omega,t) = \begin{cases} 1 \text{ if } \ \omega \in A, \\ \\ 0 \text{ if } \ \omega \in A^c. \end{cases}$$

Clearly, $\omega \to \sup_{0\le t\le 1} X(\omega,t)$ is not measurable. Hence, it does not define a random variable. In order to overcome this difficulty involving supremum and infimum, we impose the condition of separability of stochastic processes.

Definition 3.16. The process $X = \{X(t),\, t\in\mathbb{T}\}$ is said to be *separable* if there is a countable dense subset $\mathbb{S}$ of $\mathbb{T}$, called the separating set, and a set Ω_0 with $\mathbf{P}(\Omega_0) = 0$, called the negligible set, such that if $\omega\in\Omega_0^c$ and $t\in\mathbb{T}$, there is a sequence $s_n\in\mathbb{S}$, $s_n\to t$, with $X(\omega,s_n)\to X(\omega,t)$.

The following proposition is well known.

Proposition 3.12. *Every real stochastic process $X = \{X(t),\, t\in\mathbb{T}\}$ possesses a separable version. Moreover, if a separable stochastic process X is continuous in probability, then any countable dense subset in $\mathbb{T}$ is a separating set.*

Proof. See, e.g., Ash and Gardner [12].

Remark 3.5. By virtue of Proposition 3.12, we may therefore only consider separable stochastic processes.

Example 3.7. (Nonseparable stochastic process) Consider a probability space $(\Omega,\mathscr{F},\mathbf{P})$ on which is defined a positive random variable Z with continuous distribution $\mathbf{P}(Z=x)=0$ for each x. For $t\ge 0$, put $X(\omega,t)=0$ for all $\omega\in\Omega$, and put

$$Y(\omega,t) = \begin{cases} 1 \text{ if } Z(\omega)=t, \\ \\ 0 \text{ if } Z(\omega)\ne t. \end{cases}$$

Since Z has continuous distribution, $\mathbf{P}\{\omega : X(\omega,t)\ne Y(\omega,t)\} = \mathbf{P}\{\omega : Z(\omega) = t\} = 0$ for each t, and so X and Y are versions of one another. However, the stochastic process Y is not separable unless the separating set $\mathbb{S}$ contains the point $Z(\omega)$. The set of ω for which $Y(\omega,\cdot)$ is separable with respect to $\mathbb{S}$ is thus contained in $\{\omega : Z(\omega)\in\mathbb{S}\}$, a set of probability zero since $\mathbb{S}$ is countable and Z has a continuous distribution.

Definition 3.17. A *filtration* is a family $(\mathscr{F}_t)_{t\ge 0}$ of increasing sub-σ-fields of $\mathscr{F}$ (i.e. $\mathscr{F}_t\subset\mathscr{F}_s\subset\mathscr{F}$ for all $0\le t<s<\infty$). The filtration is said to be *right continuous* if $\mathscr{F}_t = \bigcap_{s>t}\mathscr{F}_s$ for all $t\ge 0$. When the probability space is complete, the filtration is said to satisfy the *usual conditions* if it is right continuous and $\mathscr{F}_0$ contains all $\mathbf{P}$-null sets.

From now on, unless otherwise specified, we shall always be working on a filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\ge 0},\mathbf{P})$, where the filtration $(\mathscr{F}_t)_{t\ge 0}$ satisfies the usual conditions.

Let $X = \{X(t),\, t\in[0,\infty)\}$ be an $\mathbb{R}$-valued stochastic process.

Definition 3.18. X is said to be *adapted* if for every t, $X(t)$ is $\mathscr{F}_t$-measurable. It is said to be *measurable* if the stochastic process regarded as a function of two variables (ω,t) from $\Omega \times [0,\infty)$ to $\mathbb{R}$ is $\mathscr{F} \times \mathscr{B}([0,\infty))$-measurable, where $\mathscr{B}([0,\infty))$ is the family of all Borel subsets of $[0,\infty)$.

Definition 3.19. Let $X = \{X(t),\, t \in [0,\infty)\}$ be a stochastic process. The *natural filtration* $\mathscr{F}_t^X = \sigma(X(s),\, 0 \le s \le t)$ of X is the smallest filtration with respect to which X is adapted.

Example 3.8. Let $([0,1], \mathscr{B}([0,1]), \lambda)$ be a probability space defined in Example 3.1 and a random variable $X(\omega) = \omega$ defined on this space. Now, consider a stochastic process $Y : \Omega \times [0,1] \to \mathbb{R}$ defined by $Y(\omega,t) = X(\omega)$. Clearly, the filtration $\mathscr{F}_t^Y$ of Y is

$$\mathscr{F}_t^Y = \sigma\Big(\bigcup_{0 \le s \le t} \sigma(Y(s))\Big) = \sigma(X)\,.$$

Because $X(\omega) = \omega$ is the identity random variable, the σ-field $\sigma(X)$ generated by the random variable X is $\mathscr{B}([0,1])$. Thus, the natural filtration of Y is $\mathscr{F}_t^Y = \mathscr{B}([0,1])$, $t \ge 0$.

Definition 3.20. The stochastic process X is said to be *progressively measurable* or *progressive* if for every $T \ge 0$, $\{X(t),\, 0 \le t \le T\}$ regarded as a function of (ω,t) from $\Omega \times [0,T]$ to $\mathbb{R}$ is $\mathscr{F}_t \times \mathscr{B}([0,T])$-measurable, where $\mathscr{B}([0,T])$ is the family of all Borel subsets of $[0,T]$.

Proposition 3.13. *If the process (X_t) is progressively measurable, then is also measurable.*

Proof. Let $B \in \mathscr{B}(\mathbb{R})$. Then

$$\begin{aligned} X^{-1}(B) &= \Big\{(\omega,s) \in \Omega \times \mathbb{R}_+ : X(\omega,s) \in B\Big\} \\ &= \bigcup_{n=0}^{\infty} \Big\{(\omega,s) \in \Omega \times [0,n] : X(\omega,s) \in B\Big\}. \end{aligned}$$

Since

$$\Big\{(\omega,s) \in \Omega \times [0,n] : X(\omega,s) \in B\Big\} \in \mathscr{F}_n \otimes \mathscr{B}([0,n])$$

for all $n \ge 0$, we have that $X^{-1}(B) \in \mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+)$.

Before giving an example of a progressively measurable stochastic process, we need the following definition.

Definition 3.21. A stochastic process $X = \{X(\omega,t),\, t \ge 0\}$ is said to be (1) *right-continuous*, if **P**-almost all of its paths $t \to X(\omega,t)$ are right-continuous, i.e., if $X(\omega,t) = \lim_{s \downarrow t} X(\omega,s)$ for all $t \in \mathbb{R}_+$,

(2) *left-continuous*, if **P**-almost all of its paths $t \to X(\omega,t)$ are left-continuous, i.e., if $X(\omega,t) = \lim_{s \uparrow t} X(\omega,s)$ for all $t \in \mathbb{R}_+$,

Example 3.9. Any right- or left-continuous adapted stochastic process $X = \{X(\omega,t), t \geq 0\}$ is progressively measurable.

To see this, let us assume that X is a right-continuous stochastic process and define the sequence of stochastic processes

$$X_{n,t}(\omega,s) = \begin{cases} X(\omega, \frac{kt}{n}) \text{ if } \frac{(k-1)t}{n} < s \leq \frac{kt}{n},\ k = 1,\ldots,n, \\ \\ X(\omega,0) \quad \text{if } \ s = 0, \end{cases}$$

for $n = 1,2,\ldots,n$ and $t \geq 0$. Now, for any $F \in \mathscr{F}$, write

$$\begin{aligned} &\Big\{(\omega,s) : \ \omega \in \Omega,\, 0 \leq s \leq t,\, X_{n,t}(\omega,s) \in F\Big\} \\ &= \Big\{\omega : X(\omega,0) \in F\Big\} \times \Big\{0\Big\} \cup \bigcup_{k=1}^{n} \Big(\Big\{\omega : X(\omega, \frac{kt}{n}) \in F\Big\} \times \Big(\frac{(k-1)t}{n}, \frac{kt}{n}\Big]\Big). \end{aligned}$$

Clearly, this set belongs to $\mathscr{F}_t \times \mathscr{B}([0,t])$ since X is adapted. Hence, for each $n \geq 1$ and $t \geq 0$, $X_{n,t}$ is $\mathscr{F}_t \times \mathscr{B}([0,t])$-measurable. By the right continuity of X we have $X_{n,t}(\omega,s) \to X(\omega,s)$ as $n \to \infty$ for all $\omega \in \Omega$ and $0 \leq s \leq t$. Since limits of measurable functions are measurable, we conclude that X is progressively measurable.

It is worth mentioning that the class of progressively measurable process is too large. Motivated by this remark, we define the so-called predictable process.

Let $\mathscr{L}$ denote the family of all real-valued functions $Y(\omega,t)$ defined on $\Omega \times \mathbb{R}_+$ which are measurable with respect to $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+)$ and have the following properties:

(i) $Y = (Y_t)$ is adapted to $(\mathscr{F}_t)$,

(ii) For each $\omega \in \Omega$, the function $t \to Y(\omega,t)$ is left-continuous.

Now, let $\mathscr{P}$ be the smallest σ-field of subsets of $\Omega \times \mathbb{R}_+$ with respect to which all the functions belonging to $\mathscr{L}$ are measurable.

Definition 3.22. A stochastic process $X = (X_t)$ is *predictable* if the function $(\omega,t) \to X(\omega,t)$ is $\mathscr{P}$-measurable. Alternatively, the predictable processes are sometimes called *previsible*.

Predictable processes are extensively used as integrands for stochastic integrals and are often not restricted to the adapted and left-continuous case.

Below are some simple examples of predictable processes.

Example 3.10. All $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+)$-measurable, adapted, and left-continuous processes are predictable.

Example 3.11. A simple process (Φ_t) which is defined to be of the form

$$\Phi(\omega,t) = \Phi_0(\omega) I_{\{0\}}(t) + \sum_{j=0}^{n-1} \Phi_j(\omega) I_{(t_j,t_{j+1}]}(t), \ \ (0 = t_0 < t_1 < \ldots < t_n) \quad (3.8)$$

is predictable if each Φ_j is $\mathscr{F}_t$-measurable.

The process given by (3.8) is adapted and left-continuous.

Example 3.12. Let η_t be an adapted, right-continuous step process given by

$$\eta(\omega,t) = \sum_{i=0}^{n} \eta(\omega,t_j)\mathbf{1}_{[t_j,t_{j+1})}(t)\,.$$

Let (Ψ_t) be the process defined by $\Psi(\omega,t) = \eta(\omega,t^-)$, the left limit of $\eta(\omega,\cdot)$. Then (Ψ_t) is predictable.

The σ-field $\mathscr{P}$ has another characterization.

Proposition 3.14. *The σ-field $\mathscr{P}$ is generated by all sets of the form*

$$A \times (s,t],\, 0 \le s < t < \infty, A \in \mathscr{F}_s \text{ or } \mathrm{A} \times \{0\},\, \mathrm{A} \in \mathscr{F}_0\,.$$

Proof. We follow the proof given in Kallianpur [107]. Denote by $\mathscr{U}$ the class of all functions of (ω,t) of the form $I_B(\omega)I_{(u,v]}(t)$, where $B \in \mathscr{F}_0$ and $u,v \in \mathbb{R}_+$ $(u \le v)$ or of the form $I_B(\omega)I_0(t)$, where $B \in \mathscr{F}_0$. Clearly, each member of $\mathscr{U}$ is $\mathscr{P}$-measurable, so that $\sigma(\mathscr{U}) \subset \mathscr{P}$, where $\sigma(\mathscr{U})$ is the smallest σ-field with respect to which all functions in $\mathscr{U}$ are measurable. To prove the converse inclusion, let $\Phi \in \mathscr{L}$. Then, for each (ω,t), Φ is the limit of a sequence of step processes of the form given in Example 3.11. Such a sequence is given by (Φ^n), where $\Phi^n(\omega,t) = \Phi_0(\omega)I_{\{0\}}(t) + \sum_{j=0}^{j_n-1} \Phi(\omega,t_j^n)I_{(t_j^n,t_{j+1}^n]}(t)$ and $0 = t_0^n < t_1^n < \ldots < t_{j_n}^n)$ is a subdivision of $[0,n]$ such that the length of each subinterval is less than or equal to $\frac{1}{n}$. Since $\Phi(\cdot,t_j^n)$ is $\mathscr{F}_{t_j^n}$-measurable, it is the pointwise limit of a sequence of step functions of the $\sum_i \alpha_i I_{B_i}(\omega)$ where $B_i \in \mathscr{F}_{t_j^n}$. Hence, $\Phi^n(\omega,t)$ is measurable with respect to $\sigma(\mathscr{U})$, which implies the measurability of $\Phi(\omega,t)$ with respect to $\sigma(\mathscr{U})$. This shows that $\mathscr{P} \subset \sigma(\mathscr{U})$ and completes the proof.

Remark 3.6. The σ-field $\mathscr{P}$ given in Proposition 3.14 is called a *predictable σ-field* and its elements are called *predictable sets*. It plays an essential role in the construction of stochastic integrals.

The following result gives the relation between predictable and progressively measurable processes.

Proposition 3.15. *Every predictable stochastic process is progressively measurable.*

Proof. For a proof, see, e.g., Meyer [141].

3.2.3 Stopping Times

In what follows we are given a filtered probability space $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbf{P})$. We are often interested in events that occur at a random time. A *random time* is simply a $[0,\infty)$-valued random variable on the probability space. A very special class of random times is the so-called class of stopping times.

More precisely, we have the following definition.

Definition 3.23. A random time τ is a *stopping time* for the filtration $(\mathscr{F}_t)_{t\geq 0}$ if $\{\tau \leq t\} \in \mathscr{F}_t$ for every $t \geq 0$.
The stopping time is said to be *finite* if $\mathbf{P}(\tau = \infty) = 0$.

Suppose τ is a stopping time for the filtration $(\mathscr{F}_t)_{t\geq 0}$. The σ-field $\mathscr{F}_\tau$ is defined to be the set of events $A \in \mathscr{F}$ such that $A \cap \{\tau \leq t\} \in \mathscr{F}_t$ for every $t \geq 0$. $\mathscr{F}_\tau$ can be viewed as the set of events determined prior to the stopping time τ.

Example 3.13. Any positive constant is a stopping time.

To see this, let $\tau = a$, where a is a nonnegative number. Then, for $t \geq 0$,

$$\{\omega \in \Omega : \tau(\omega) \leq t\} = \begin{cases} \emptyset \ \text{ if } \ a > t, \\ \\ \Omega \ \text{if } \ a \leq t. \end{cases}$$

Hence, for $t \geq 0$, $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathscr{F}_t$.

Example 3.14. If $X = \{X_n,\, n \geq 0\}$ is a sequence of real-valued random variables and $\mathscr{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, the hitting time

$$\tau(\omega) = \inf\{n \geq 0 :\, X_n(\omega) > a\},\ a \in \mathbb{R}\,,$$

is an $\mathscr{F}_n$-stopping time.

To prove this, observe that

$$\Big\{\tau \leq n\Big\} = \Big\{X_0 > a\Big\} \bigcup_{k=1}^{n} \Big\{X_0 \leq a, \ldots, X_{k-1} \leq a, X_k > a\Big\}\,.$$

Now, note that $\Big\{\tau \leq n\Big\}$ consists of finite intersections and unions of events in $\mathscr{F}_k$ and that $\mathscr{F}_k \subset \mathscr{F}_n$ for $k \leq n$. Thus, $\Big\{\tau \leq n\Big\} \in \mathscr{F}_n$.

Stopping time has some nice properties.

Proposition 3.16. *(i) If τ_1 and τ_2 are stopping times, then $\tau_1 \wedge \tau_2 = \inf\{\tau_1, \tau_2\}$ and $\tau_1 \vee \tau_2 = \sup\{\tau_1, \tau_2\}$ are also stopping times.*

(ii) If τ is a stopping time and $a \in [0, \infty)$, then $\tau \wedge a$ is also a stopping time.

(iii) If τ is a finite stopping time, then it is $\mathscr{F}_\tau$-measurable.

(iv) If τ_1 and τ_2 are stopping times and $\tau_1 \leq \tau_2$, then $\mathscr{F}_{\tau_1} \subset \mathscr{F}_{\tau_2}$.

Proof. See, e.g., Métivier [140].

3.2.4 Gaussian Processes

Definition 3.24. The real-valued stochastic process $X = \{X(t),\, t \in \mathbb{T}\}$ is called a *Gaussian process* if, for any finite subset $F \subset \mathbb{T}$, the random vector $X_F := \{X(t), t \in$

$F\}$ has multivariate Gaussian distribution, with probability density

$$f_F(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det\Sigma}} \exp\left\{ -\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\right\},$$

with parameters $\mu \in \mathbb{R}^n$ and Σ.

(Here, $\mathbf{y}'$ denotes the transpose of the vector $\mathbf{y}$.) The quantity Σ is a symmetric positive-definite $n \times n$ matrix, Σ^{-1} is its inverse, and $\det\Sigma$ its determinant.
Equivalently, X is Gaussian if every finite linear combination $\sum_{t\in F} X(t)$ has a Gaussian distribution on $\mathbb{R}$. The covariance function of Gaussian process X is the bivariate function

$$R(s,t) = \mathrm{Cov}(X(s),X(t)) = \mathbf{E}\Big[\big(X(s)-\mathbf{E}X(s)\big)\big(X(t)-\mathbf{E}X(t)\big)\Big].$$

Like the Gaussian vector, it is important to note that the mean function and covariance function of a Gaussian process completely determine all of the finite-dimensional distributions.

Example 3.15. Let $Z = (Z_1,\ldots,Z_m) \in \mathbb{R}^m$ be a Gaussian vector. Define X as follows:

$$X(t) = \sum_{k=1}^{m} Z_k w_k(t),\ t \geq 0$$

where $w_k(t)$, $k = 1,\ldots,m$ are real-valued, deterministic, and continuous functions. We claim X is a Gaussian process. Indeed, let $X_n = (X(t_1),\ldots,X(t_n))$, where $n \geq 1$ is an integer and $(t_1,\ldots,t_n)$ denote arbitrary elements in $[0,\infty)$. The vector X_n can be expressed as a linear transformation of the Gaussian vector Z so that it is Gaussian.

3.2.5 Martingales

In this subsection, we introduce and study a very important class of stochastic processes: the so-called martingales. Martingales arise naturally in many branches of the theory of stochastic processes. In particular, they play a key role in the study of the Brownian motion. They are also crucial for the understanding of the Itô integrals. Indefinite Itô integrals are constructed in such a way that they constitute martingales. Throughout this subsection, the index set $\mathbb{T}$ denotes an arbitrary interval of $\mathbb{R}_+$.

Definition 3.25. The stochastic process $X = \{X(t),\, t \in \mathbb{T}\}$ is called a continuous-time martingale with respect to the filtration $(\mathscr{F}_t,\, t \in \mathbb{T})$, we write $(X,(\mathscr{F}_t))$, if

(i) $\mathbf{E}|X(t)| < \infty$ for all $t \in \mathbb{T}$;
(ii) X is adapted to $\{\mathscr{F}_t\}$;
(iii)

$$\mathbf{E}\big[X(t)\,|\,\mathscr{F}_s\big] = X(s)\ \ \mathbf{P}-\text{a.s.} \tag{3.9}$$

for all $s < t$ in $\mathbb{T}$.

It follows from the definition of conditional expectations that the identity (3.9) is equivalent to the statement

$$\int_F X(t)\,d\mathbf{P} = \int_F X(s)\,d\mathbf{P}, \text{ for } F \in \mathscr{F}_s,\ 0 \le s \le t$$

and that the expectation function $\mathbf{E}X$ is constant (that is, $\mathbf{E}(X(s)) = \mathbf{E}(X(t))$ for all s and t).

When equality is substituted with $\le$, the process is called *supermartingale*. When it is substituted with $\ge$, the process is called *submartingale*.

It is also possible to define a discrete-time martingale $X = \{X_n,\, n = 0,1,2,\dots\}$. In this case, the property (3.9) becomes

$$\mathbf{E}\big[X_{n+k} \mid \mathscr{F}_n\big] = X_n\,, \quad k \ge 0\,.$$

The basic properties of conditional expectations give us the following properties of a martingale.

Proposition 3.17. *Let X be an integrable random variable and $(\mathscr{F}_t)_{t\in\mathbb{T}}$ a filtration. For $t \in \mathbb{T}$, define $M(t) = \mathbf{E}\big[X \mid \mathscr{F}_t\big]$. Then $M = \{M(t),\, t \in \mathbb{T}\}$ is an $(\mathscr{F}_t)$-martingale and M is uniformly integrable. In addition, if φ is a convex function such that $\mathbf{E}\big|\varphi(M(t))\big| < \infty$ for all $t \in \mathbb{T}$, then the stochastic process $\varphi(M)$ is a submartingale.*

Proof. By the Jensen inequality for conditional expectation (Proposition 3.10 (vii)) and the integrability of X, we have

$$\mathbf{E}\big|M(t)\big| \le \mathbf{E}\Big[\mathbf{E}\big[|X| \,|\mathscr{F}_t\big]\Big] = \mathbf{E}\big|X\big| < \infty\,.$$

Also, M is $\mathscr{F}_t$-adapted because $\mathbf{E}\Big[X \mid \mathscr{F}_t\Big]$ is $\mathscr{F}_t$-measurable for each $t \ge 0$. Properties of the conditional expectation give

$$\mathbf{E}\Big[M(t) \mid \mathscr{F}_s\Big] = \mathbf{E}\Big[\mathbf{E}\big[X(t) \mid \mathscr{F}_t\big] \mid \mathscr{F}_s\Big] = \mathbf{E}\big[X(t) \mid \mathscr{F}_s\big] = M(s)$$

for all $s \le t$. Hence, M obeys the properties of a martingale.

Similarly, the Jensen inequality applied to a convex function φ and properties of conditional expectation yield

$$\mathbf{E}\Big[\varphi(M(t)) \mid \mathscr{F}_s\Big] \ge \varphi\Big(\mathbf{E}\big[\mathbf{E}\big[X \mid \mathscr{F}_t\big] \mid \mathscr{F}_s\big]\Big) = \varphi\Big(\mathbf{E}\big[X \mid \mathscr{F}_s\big]\Big) = \varphi(M(s))\,.$$

Thus, $\varphi(M)$ is a submartingale.

Let us now define a Brownian motion which plays a key role in the construction of stochastic integrals.

Definition 3.26. A (standard one-dimensional) *Brownian motion* is a continuous adapted real-valued process $(\mathbb{B}(t),\, t \ge 0)$ such that

(i) $\mathbb{B}(0) = 0$;
(ii) $\mathbb{B}(t) - \mathbb{B}(s)$ is independent of $\mathscr{F}_s$ for all $0 \le s < t$;
(iii) $\mathbb{B}(t) - \mathbb{B}(s)$ is $\mathscr{N}(0, t-s)$-distributed for all $0 \le s \le t$.

Note that the Brownian motion $\mathbb{B}$ has the following properties:

(a) $\mathbb{B}$ has independent increments, that is, for $t_1 < t_2 < \cdots < t_n$, $\mathbb{B}(t_1) - \mathbb{B}(0), \mathbb{B}(t_2) - \mathbb{B}(t_1), \ldots, \mathbb{B}(t_n) - \mathbb{B}(t_{n-1})$ are independent random variables;

(b) $\mathbb{B}$ has stationary increments, that is, $\mathbb{B}(t+s) - \mathbb{B}(t)$ has the same distribution as $\mathbb{B}(s) - \mathbb{B}(0)$.

The following proposition gathers some simple examples of stochastic processes which have the martingale property.

Proposition 3.18. *Let* $\{\mathbb{B}(t), t \ge 0\}$ *be a Brownian motion, and define* $\mathscr{F}_t = \sigma\big(\mathbb{B}(s); s \le t\big)$. *Then the following stochastic processes are martingale with respect to the same filtration:*

(i) $(\mathbb{B}(t), \mathscr{F}_t)_{t\ge 0}$ *itself;*

(ii) $(\mathbb{B}(t)^2 - t, \mathscr{F}_t)_{t\ge 0}$;

(iii) for every $\theta \in \mathbb{R}$, *the process* $\left(\exp\left[\theta\mathbb{B}(t) - \frac{\theta^2}{2}t\right], \mathscr{F}_t\right)_{t\ge 0}$ *(called an exponential martingale).*

Proof. Let us first verify that $\big(\mathbb{B}(t), \mathscr{F}_{t\ge 0}\big)_{t\ge 0}$ is a martingale. Since $\mathbb{B}(t) \sim \mathscr{N}(0,t)$, $\mathbb{B}(t)$ is clearly integrable, and second, since $\mathbb{B}(t) - \mathbb{B}(s)$ is independent of $\mathbb{B}(s)$ by Definition 3.26(ii), $\mathbf{E}\Big[\mathbb{B}(t) - \mathbb{B}(s) \,|\, \mathscr{F}_s\Big] = 0$, equivalently, $\mathbf{E}\Big[\mathbb{B}(t) \,|\, \mathscr{F}_s\Big] = \mathbb{B}(s)$.

Likewise, using the properties of conditional expectation,

$$\begin{aligned}\mathbf{E}\Big[\big(\mathbb{B}(t) - \mathbb{B}(s)\big)^2 \,|\, \mathscr{F}_s\Big] &= \mathbf{E}\Big[\mathbb{B}(t)^2 - 2\mathbb{B}(t)\mathbb{B}(s) + \mathbb{B}(s)^2 \,|\, \mathscr{F}_s\Big] \\ &= \mathbf{E}\Big[\mathbb{B}(t)^2 \,|\, \mathscr{F}_s\Big] - \mathbb{B}(s)^2 . \end{aligned} \tag{3.10}$$

On the other hand, since $\mathbb{B}(t) - \mathbb{B}(s) \sim \mathscr{N}(0, t-s)$ is independent of $\mathbb{B}(s)$,

$$\mathbf{E}\Big[\big(\mathbb{B}(t) - \mathbb{B}(s)\big)^2 \,|\, \mathscr{F}_s\Big] = \mathbf{E}\Big[\big(\mathbb{B}(t) - \mathbb{B}(s)\big)^2\Big] = t - s . \tag{3.11}$$

Hence, combining (3.10) and (3.11) we obtain

$$\mathbf{E}\Big[\mathbb{B}(t)^2 - t \,|\, \mathscr{F}_s\Big] = \mathbb{B}(s)^2 - s .$$

This concludes that $\mathbb{B}(t)^2 - t$ is an $(\mathscr{F}_t)$-martingale.

As to part (iii), since $\mathbb{B}(t) - \mathbb{B}(s) \sim \mathscr{N}(0, t-s)$, its moment-generating function is

$$\mathbf{E}\Big[\exp\big[\theta(\mathbb{B}(t) - \mathbb{B}(s))\big]\Big] = \exp\left[\frac{1}{2}\theta^2(t-s)\right],$$

for any $\theta \in \mathbb{R}$ and $0 \le s \le t$.
Then, using the fact that $\mathbb{B}(t) - \mathbb{B}(s)$ is independent of $\mathscr{F}_s$,

$$\begin{aligned}\mathbf{E}\Big[\exp\big[\theta\mathbb{B}(t)-\frac{1}{2}\theta^2t\big]\mid\mathscr{F}_s\Big] &= \mathbf{E}\Big[\exp\big[\theta(\mathbb{B}(t)-\mathbb{B}(s))+\theta\mathbb{B}(s)-\frac{1}{2}\theta^2t\big]\mid\mathscr{F}_s\Big]\\ &= \exp\big[\theta\mathbb{B}(s)-\frac{1}{2}\theta^2t\big]\,\mathbf{E}\Big[\exp\big[\theta(\mathbb{B}(t)-\mathbb{B}(s))\big]\Big]\\ &= \exp\big[\theta\mathbb{B}(s)-\frac{1}{2}\theta^2t\big]\,\exp\big[\frac{1}{2}\theta^2(t-s)\big]\\ &= \exp\big[\theta\mathbb{B}(s)-\frac{1}{2}\theta^2s\big]\,.\end{aligned}$$

Hence, the process $\left(\exp\left[\theta\mathbb{B}(t)-\frac{\theta^2}{2}t\right],\mathscr{F}_t\right)_{t\geq 0}$ is a martingale.

For continuous martingales we have the following inequalities due to Doob.

Theorem 3.3. (Doob's inequality) *Let* $\{M(t)\}_{0\leq t\leq T}$ *be a continuous martingale.*
(i) If $p\geq 1$ *and* $M(t)\in L^p(\Omega;\mathbb{R})$*, then*

$$\mathbf{P}\Big\{\omega:\sup_{0\leq t\leq T}\big|M(\omega,t)\big|>c\Big\}\leq\frac{\mathbf{E}\big|M(T)\big|^p}{c^p}\,;$$

(ii) If $p>1$ *and* $M(t)\in L^p(\Omega;\mathbb{R})$*, then*

$$\mathbf{E}\Big[\sup_{0\leq t\leq T}M(t)^p\Big]\leq\Big(\frac{p}{p-1}\Big)^p\mathbf{E}\Big[M(T)^p\Big]\,.$$

Further discussions on this topic may be found in Stroock and Varadhan [168] or Revuz and Yor [158].

3.3 Stochastic Integrals in One Dimension

3.3.1 Motivation

In applications, it is typical to characterize the current state of a physical system by a real function of time $x(t)$, $t\geq 0$, called the state. Generally, the behavior of a physical system based on an input $w(t)$ for $t\geq 0$, can be specified by a differential equation of the form

$$\frac{dx(t)}{dt}=\mu\left(x(t)\right)+\sigma(x(t))w(t),\; t\geq 0\,, \tag{3.12}$$

where the functions μ and σ depend on the system properties. In classical analysis, the study of the solutions of such an equation is based on the assumptions that the system properties and the input are perfectly known and deterministic.

Here, we generalize Eq. (3.12) by assuming that the input is a real stochastic process. Because the input is random, the state becomes a real stochastic process.

Now, let X denote the solution of (3.12) with w replaced by a stochastic process Z. It is customary to assume that Z is a "white noise" process for which $\mathbf{E}\big[Z(t)\big] = 0$ and $\mathrm{Cov}(Z(s), Z(t)) = 1$ if $s = t$ and is zero otherwise. It is important to note that for $t_1 < t_2 < t_3$,

$$\mathrm{Cov}\Big[\int_{t_1}^{t_2} Z(s)\,ds,\ \int_{t_2}^{t_3} Z(s)\,ds\Big] = 0 \tag{3.13}$$

whereas

$$\mathrm{Var}\Big[\int_0^t Z(s)\,ds\Big] = t\,. \tag{3.14}$$

The Gaussian white noise process is often used. Such a stochastic process $\big\{Z(t), t \in \mathbb{R}\big\}$ has irregular sample paths and is very difficult to work with directly. As a result, it is easier to work with its integral. This suggests writing (3.12) in the form

$$X(t) = X(0) + \int_0^t \mu(X(s))\,ds + \int_0^t \sigma(X(s))Z(s)\,ds\,. \tag{3.15}$$

In this integrated version, we need to make mathematical sense of the stochastic integral involving the integrator $Z(s)\,ds$. From a notational standpoint, it is common to write

$$dX(t) = \mu\,(X(t))\,dt + \sigma(X(t))Z(t)\,dt\,. \tag{3.16}$$

Note that given a Brownian motion $\mathbb{B}$, it is not difficult to verify that

$$\mathrm{Cov}\Big[\mathbb{B}(t_2) - \mathbb{B}(t_1), \mathbb{B}(t_3) - \mathbb{B}(t_2)\Big] = 0 \text{ and } \mathrm{Var}\Big[\mathbb{B}(t) - \mathbb{B}(0)\Big] = t\,.$$

Given the similarity with (3.13) and (3.14), the latter hints that $\mathbb{B}$ can be viewed as integrated white noise so that we can rigorously define $\int_0^t Z(t)\,dt$ to be $\mathbb{B}(t)$. This is quite an oversimplification. To write $\mathbb{B}(t) = \int_0^t Z(t)\,dt$ would require that $\mathbb{B}$ is differentiable almost everywhere (in time t). Unfortunately, this is not the case: $\mathbb{B}$ is non differentiable at t. This oversimplification comes from the fact that white noise does not exist as a well-defined stochastic process. On the other hand, Brownian motion is well defined, so this suggests that we should replace (3.15) with

$$X(t) = x_0 + \int_0^t \mu\,(X(s))\,ds + \int_0^t \sigma(X(s))d\mathbb{B}(s) \tag{3.17}$$

and (3.16) with

$$\begin{cases} dX(t) = \mu\,(X(t))\,dt + \sigma(X(t))\,d\mathbb{B}(t) \\ \\ X(0) = x_0. \end{cases} \tag{3.18}$$

Note that in (3.17), the integral $\int_0^t \mu(X(s))\,ds$ can be defined via a standard Riemann approximation. On the other hand, $\int_0^t \sigma(X(s))d\mathbb{B}(s)$ must be defined differently since the integrator is a non differentiable stochastic process.

This leads us to outline the construction of the so-called Itô integral.

3.3.2 Itô Integrals

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ be a filtered probability space and $\mathbb{B} = \left\{\mathbb{B}(t),\ t \geq 0\right\}$ be a one-dimensional Brownian motion defined on this space.

Definition 3.27. Let $0 \leq S < T < \infty$. Denote by $\mathscr{V}([S,T];\mathbb{R})$ the space of all real-valued measurable $(\mathscr{F}_t)$-adapted processes $\Phi = \left\{\Phi(t),\ t \geq 0\right\}$ such that

$$\left\|\Phi\right\|_{\mathscr{V}}^2 = \mathbf{E}\Big[\int_S^T \left|\Phi(t)\right|^2 dt\Big] < \infty .$$

We identify Φ and $\bar{\Phi}$ in $\mathscr{V}([S,T];\mathbb{R})$ if $\left\|\Phi - \bar{\Phi}\right\|_{\mathscr{V}}^2 = 0$. In this case we say that Φ and $\bar{\Phi}$ are *equivalent* and we write $\Phi = \bar{\Phi}$.

It is routine to show that the space $\mathscr{V}([S,T];\mathbb{R})$ equipped with the norm $\left\|\cdot\right\|_{\mathscr{V}}$ is a Banach space. Furthermore, without loss of generality we may assume that every stochastic process $\Phi \in \mathscr{V}([S,T];\mathbb{R})$ is predictable.

Since full details on the construction of the Itô integral $\displaystyle\int_S^T \Phi(t)\, d\mathbb{B}(t)$ for stochastic processes $\Phi \in \mathscr{V}([S,T];\mathbb{R})$ can be found in either Øksendal [150] or Mao and Yuan [135], here we shall outline only its construction. The idea of construction is as follows. First define the integral $\displaystyle\int_S^T \Psi(t)\, d\mathbb{B}(t)$ for a class of simple processes Ψ. Then we show that each $\Phi \in \mathscr{V}([S,T];\mathbb{R})$ can be approximated by such simple processes Ψ's and we define the limit of $\displaystyle\int_S^T \Psi(t)\, d\mathbb{B}(t)$ as the integral $\displaystyle\int_S^T \Phi(t)\, d\mathbb{B}(t)$.

Let us first introduce the concept of simple stochastic processes.

Definition 3.28. A stochastic process $\Psi \in \mathscr{V}([S,T];\mathbb{R})$ is called *simple* if it is of the form

$$\Psi(\omega,t) = \alpha_0(\omega)\,\mathbf{1}_{[t_0,t_1)}(t) + \sum_{i=0}^{k-1} \alpha_i(\omega)\,\mathbf{1}_{(t_i,t_{i+1}]}(t)\,,$$

with a partition $S = t_0 < t_1 < \ldots < t_k = T$ of $[S,T]$ and bounded $\mathscr{F}_{t_i}$-measurable random variables α_i, $0 \leq i \leq k-1$.

For any simple stochastic process $\Psi \in \mathscr{V}([S,T];\mathbb{R})$ we define

$$\int_S^T \Psi(t)\, d\mathbb{B}(t) := \sum_{i=0}^{k-1} \alpha_i[\mathbb{B}(t_{i+1}) - \mathbb{B}(t_i)]\,. \tag{3.19}$$

Obviously, the integral $\int_S^T \Psi(t)\, d\mathbb{B}(t)$ is a well-defined random variable. Moreover, the following properties hold:

$$\mathbf{E}\Big[\int_S^T \Psi(t)\,d\mathbb{B}(t)\Big] = 0\,, \tag{3.20}$$

$$\mathbf{E}\Big|\int_S^T \Psi(t)\,d\mathbb{B}(t)\Big|^2 = \int_S^T \mathbf{E}\big|\Psi(t)\big|^2\,dt\,. \tag{3.21}$$

To prove these identities, note that α_i is $\mathscr{F}_{t_i}$-measurable and that $\mathbb{B}(t_{i+1})-\mathbb{B}(t_i)$ is independent of $\mathscr{F}_{t_i}$. Hence,

$$\begin{aligned}\mathbf{E}\int_S^T \Psi(t)\,d\mathbb{B}(t) &= \sum_{i=0}^{k-1}\mathbf{E}\Big[\alpha_i\big[\mathbb{B}(t_{i+1})-\mathbb{B}(t_i)\big]\Big]\\ &= \sum_{k=0}^{k-1}\mathbf{E}(\alpha_i)\mathbf{E}\big[\mathbb{B}(t_{i+1})-\mathbb{B}(t_i)\big] = 0\,.\end{aligned}$$

Moreover, note that $\mathbb{B}(t_{j+1})-\mathbb{B}(t_j)$ is independent of $\alpha_i\alpha_j(\mathbb{B}(t_{i+1})-\mathbb{B}(t_i))$ if $i<j$. Thus,

$$\begin{aligned}\mathbf{E}\Big|\int_S^T \Psi(t)\,d\mathbb{B}(t)\Big|^2 &= \sum_{0\le i,\,j\le k-1}^{k-1}\mathbf{E}\Big[\alpha_i\alpha_j\big(\mathbb{B}(t_{i+1})-\mathbb{B}(t_i)\big)\big(\mathbb{B}(t_{j+1})-\mathbb{B}(t_j)\big)\Big]\\ &= \sum_{k=0}^{k-1}\mathbf{E}\Big[\alpha_i^2\mathbf{E}\big(\mathbb{B}(t_{i+1})-\mathbb{B}(t_i)\big)\Big]\\ &= \sum_{i=0}^{k-1}\mathbf{E}(\alpha_i^2)\mathbf{E}\big(\mathbb{B}(t_{i+1})-\mathbb{B}(t_i)\big)^2\\ &= \sum_{k=0}^{k-1}\mathbf{E}(\alpha_i^2)(t_{i+1}-t_i) = \mathbf{E}\Big[\int_S^T \big|\Psi(t)\big|^2\Big]\,.\end{aligned}$$

Also, for any simple stochastic processes $\Psi_1,\,\Psi_2\in\mathscr{V}([S,T];\mathbb{R})$ and $c_1,\,c_2\in\mathbb{R}$, we have

$$\int_S^T \big[c_1\Psi_1(t)+c_2\Psi_2(t)\big]\,d\mathbb{B}(t) = c_1\int_S^T \Psi_1(t)\,d\mathbb{B}(t)+c_2\int_S^T \Psi_2(t)\,d\mathbb{B}(t)\,. \tag{3.22}$$

The proof of (3.22) is left to the reader as an exercise.

We can now extend the Itô integral from simple stochastic processes to stochastic processes in $\mathscr{V}([S,T];\mathbb{R})$. This is based on the following approximation result.

Lemma 3.2. *For any $\Phi\in\mathscr{V}([S,T];\mathbb{R})$, there exists a sequence (Ψ_n) of simple stochastic processes such that*

$$\lim_{n\to\infty}\int_S^T \mathbf{E}\big|\Phi(t)-\Psi_n(t)\big|^2\,dt = 0\,.$$

We are now prepared to outline the construction of the Itô integral for a stochastic process $\Phi\in\mathscr{V}([S,T];\mathbb{R})$. By Lemma 3.2, there is a sequence (Ψ_n) of simple stochastic processes such that

$$\lim_{n\to\infty}\int_S^T \mathbf{E}\big|\Phi(t)-\Psi_n(t)\big|^2\,dt=0\,.$$

Thus, by property (3.20),

$$\mathbf{E}\Big|\int_S^T \Psi_n(t)\,d\mathbb{B}(t)-\int_S^T \Psi_m(t)d\mathbb{B}(t)\Big|^2=\int_S^T \mathbf{E}\big|\Psi_n(t)-\Psi_m(t)\big|^2\,dt\to 0 \text{ as } m,n\to\infty\,.$$

Hence, the sequence $\left\{\int_S^T \Psi_n(t)\,d\mathbb{B}(t),\, n\geq 1\right\}$ is a Cauchy sequence in $L^2(\Omega;\mathbb{R})$ which, in turn, implies that it is convergent.

This leads us to the following definition.

Definition 3.29. Let $\Phi\in\mathscr{V}([S,T];\mathbb{R})$. The Itô integral Φ with respect to $(\mathbb{B}(t))$ is defined by

$$\int_S^T \Phi(t)\,d\mathbb{B}(t)=\lim_{n\to\infty}\int_S^T \Psi_n(t)\,d\mathbb{B}(t)\ \text{ in } L^2(\Omega,\mathbb{R})\,,$$

where (Ψ_n) is a sequence of simple stochastic processes such that

$$\lim_{n\to\infty}\mathbf{E}\Big[\int_S^T \big|\Phi(t)-\Psi_n(t)\big|^2\,dt\Big]=0\,.$$

It is important to note that this integral does not depend on the choice of approximating sequence.

We now gather the main properties of the Itô integral.

Proposition 3.19. *Let Φ, Ψ be stochastic processes in $\mathscr{V}([S,T];\mathbb{R})$, and let $0\leq S<U<T$. Then*

(a) $\mathbf{E}\Big[\int_S^T \Phi(t)\,d\mathbb{B}(t)\Big]=0$*;*

(b) $\mathbf{E}\Big|\int_S^T \Phi(t)\,d\mathbb{B}(t)\Big|^2=\int_S^T \mathbf{E}\big|\Phi(t)\big|^2\,dt$ *(Itô Isometry);*

(c) $\int_S^T (c\Phi(t)+\Psi(t))\,d\mathbb{B}(t)=c\int_S^T \Phi(t)\,d\mathbb{B}(t)+\int_S^T \Psi(t)\,d\mathbb{B}(t)$ *(c constant);*

(d) $\int_S^T \Phi(t)\,d\mathbb{B}(t)=\int_S^U \Phi(t)\,d\mathbb{B}(t)+\int_U^T \Phi(t)\,d\mathbb{B}(t)$*;*

(e) $\int_S^T \Phi(s)\,d\mathbb{B}(s)$ *is $\mathscr{F}_T$-measurable.*

The proof is left to the reader as an exercise.

Definition 3.30. Let $\Phi\in\mathscr{V}([0,T];\mathbb{R})$. Define

$$I(t):=\int_0^t \Phi(s)\,d\mathbb{B}(s),\ \text{ for } 0\leq t\leq T\,,$$

where, by definition, $I(0)=0$. We call $I(t)$ the indefinite Itô integral of Φ.

The indefinite integral $I(t)$ has the following interesting properties.

Proposition 3.20. *The following properties hold.*

(i) $I(t)$ is $\mathscr{F}_t$-adapted and square-integrable;

(ii) $I(t), t \geq 0\Big\}$ is an $\mathscr{F}_t$-martingale and

$$\mathbf{E}\Big[\sup_{0\leq t\leq T}\big|I(t)\big|^2\Big] \leq 4\int_0^T \mathbf{E}\big|\Phi(t)\big|^2\,ds; \tag{3.23}$$

(iii) $\Big\{I(t), 0 \leq t \leq T\Big\}$ has a continuous version.

Proof. Clearly, for each t in $[0,T]$, $I(t)$ is $\mathscr{F}_t$-adapted and square-integrable. To prove part (ii), we fix $0 \leq s < t \leq T$ and use the properties of conditional expectation and Brownian motion to obtain

$$\begin{aligned}\mathbf{E}\Big(I(t)\,|\,\mathscr{F}_s\Big) &= \mathbf{E}\Big[\int_0^s \Phi(r)\,d\mathbb{B}(r)\,|\,\mathscr{F}_s\Big] + \mathbf{E}\Big[\int_s^t \Phi(r)\,d\mathbb{B}(r)\,|\,\mathscr{F}_s\Big]\\ &= \mathbf{E}\Big(I(s)\,|\,\mathscr{F}_s\Big) + \mathbf{E}\Big(\int_s^t \Phi(r)\,d\mathbb{B}(r)\,|\,\mathscr{F}_s\Big) = I(s)\,.\end{aligned}$$

The inequality (3.23) follows from Doob's martingale inequality.

As to (iii), let (Ψ_n) be a sequence of simple stochastic processes such that

$$\lim_{n\to\infty}\int_0^T \mathbf{E}\big|\Phi(s)-\Psi_n(s)\big|^2\,ds = 0\,.$$

Note from the continuity of the Brownian motion that the indefinite integrals

$$I_n(t) = \int_0^t \Psi_n(s)\,d\mathbb{B}(s),\ \ 0\leq t\leq T$$

are continuous. By Proposition 3.20(ii), the stochastic process $\Big\{I_n(t)-I_m(t), t\in 0\leq t\leq T\Big\}$ is a martingale for each pair of integers n, m. Hence, by Doob's martingale inequality, it follows that for any $\varepsilon > 0$,

$$\begin{aligned}&\mathbf{P}\Big\{\omega : \sup_{0\leq t\leq T}\big|I_n(\omega,t)-I_m(\omega,t)\big| \geq \varepsilon\Big\}\\ &\quad\leq \frac{1}{\varepsilon^2}\mathbf{E}\big|I_n(T)-I_m(T)\big|^2\\ &\quad= \frac{1}{\varepsilon^2}\int_0^T \mathbf{E}\big|\Psi_n(s)-\Psi_m(s)\big|^2\,ds \to 0 \text{ as } n,\, m\to\infty\,.\end{aligned}$$

Hence, we may choose a subsequence $n_k \uparrow \infty$ such that

$$\mathbf{P}\Big\{\omega : \sup_{0\leq t\leq T}\big|I_{n_{k+1}}(\omega,t)-I_{n_k}(\omega,t)\big| \geq 2^{-k}\Big\} \leq 2^{-k}\,.$$

By the Borel–Cantelli lemma, we have

$$\mathbf{P}\Big\{\omega : \sup_{0\le t\le T} \big|I_{n_{k+1}}(\omega,t) - I_{n_k}(\omega,t)\big| \ge 2^{-k} \text{ for infinitely many } k\Big\} = 0\,.$$

That is, there exists a set $\Omega_0 \in \mathscr{F}$ with $\mathbf{P}(\Omega_0) = 0$ and a positive integer $k(\omega)$ such that for every $\omega \in \Omega_0^c$,

$$\sup_{0\le t\le T} \big|I_{n_{k+1}}(\omega,t) - I_{n_k}(\omega,t)\big| \le 2^{-k} \text{ for } k \ge k(\omega)\,.$$

Therefore, $(I_{n_k}(\omega,\cdot))$ is uniformly convergent on $[0,T]$ for each $\omega \in \Omega_0^c$ and the limit, denoted by $J(\omega,t)$, is continuous in $t \in [0,T]$. Since $I_{n_k}(\cdot,t) \to I(\cdot,t)$ for all t, we must have

$$I(t) = J(t) \text{ a.s. for all } t \in [0,T]\,.$$

3.3.3 Itô Integrals with Stopping Time

Let τ be a stopping time and define

$$\mathbf{1}_{[0,\tau]}(t) = \begin{cases} 1 \text{ if } \ t \le \tau, \\ 0 \text{ if } \ t > \tau, \end{cases}$$

the indicator function of $[0,\tau]$. Then, the stochastic process $\Big\{\mathbf{1}_{[0,\tau]}(t),\, t \ge 0\Big\}$ is $\mathscr{F}_t$-adapted. Indeed, for each $t \ge 0$,

$$\Big\{\omega : \mathbf{1}_{[0,\tau(\omega)]}(t) \le a\Big\} = \begin{cases} \emptyset & \text{if } \ a < 0, \\ \Big\{\omega : \tau(\omega) \le t\Big\} \in \mathscr{F}_t & \text{if } \ 0 \le a < 1, \\ \Omega \in \mathscr{F}_t & \text{if } \ a \le 1. \end{cases}$$

Thus, $\Big\{\mathbf{1}_{[0,\tau]}(t),\, t \ge 0\Big\}$ is $\mathscr{F}_t$-adapted. It is also predictable.

We can now define the stochastic integrals with stopping time.

Definition 3.31. Let $\Phi \in \mathscr{V}([0,T];\mathbb{R})$ and let τ be an $\mathscr{F}_t$-stopping time such that $0 \le \tau \le T$. Define

$$\int_0^\tau \Phi(s)\, d\mathbb{B}(s) := \int_0^T \mathbf{1}_{[0,\tau]}(s)\Phi(s)\, d\mathbb{B}(s)\,. \tag{3.24}$$

Furthermore, if σ is another stopping time with $0 \le \sigma \le \tau$, we define

$$\int_\sigma^\tau \Phi(s)\, d\mathbb{B}(s) = \int_0^\tau \Phi(s)\, d\mathbb{B}(s) - \int_0^\sigma \Phi(s)\, d\mathbb{B}(s)\,. \tag{3.25}$$

Note that the integral (3.24) is well defined because the stochastic process

$\left\{\mathbf{1}_{[0,\tau]}(t)\Phi(t), t\in[0,T]\right\}$ belongs to $\mathscr{V}([0,T];\mathbb{R})$. Hence, the integral (3.25) can be rewritten as follows:

$$\int_\sigma^\tau \Phi(s)\,d\mathbb{B}(s) := \int_0^T \mathbf{1}_{[\sigma,\tau]}(s)\Phi(s)\,d\mathbb{B}(s)\,. \tag{3.26}$$

The following properties can be deduced easily from Proposition 3.19.

Proposition 3.21. *Let $\Phi\in\mathscr{V}([0,T];\mathbb{R})$ and let σ, τ be two stopping times such that $0\le\sigma\le\tau\le T$. Then*
(i) $\mathbf{E}\left[\int_\sigma^\tau \Phi(s)\,d\,\mathbb{B}(s)\right]=0,$
(ii) $\mathbf{E}\left|\int_\sigma^\tau \Phi(s)\,d\mathbb{B}(s)\right|^2=\mathbf{E}\left[\int_\sigma^\tau |\Phi(s)|^2\,ds\right].$

3.3.4 Itô Formula

In the last two subsections we defined the Itô integral of the form $\int_0^t \Phi(s)\,d\mathbb{B}(s)$ and collected its properties. However, with the exception of simple processes we do not have tools to calculate Itô integrals and to proceed some simple operations on them. It is now our objective to provide such a tool like the itô formula. Here, we present two versions of the Itô lemma.

In what follows, we use the following notation for the partial derivatives of f:

$$f_i(t,x) = \frac{\partial}{\partial x_i} f(x_1,x_2)\Big|_{x_1=t,x_2=x} \quad i=1,\,2$$
$$f_{ij}(t,x) = \frac{\partial}{\partial x_i\partial x_j} f(x_1,x_2)\Big|_{x_1=t,x_2=x} \quad i,\,j=1,\,2\,.$$

Theorem 3.4. (Version I of Itô formula) *Let $f(t,x)$ be a function whose second-order partial derivatives exist and are continuous. Then*

$$\begin{aligned} &f(t,\mathbb{B}(t))-f(s,\mathbb{B}(s))\\ &\quad=\int_s^t \left[f_1(\sigma,\mathbb{B}(\sigma))+\frac{1}{2}f_{22}(\sigma,\mathbb{B}(\sigma))\right]d\sigma+\int_s^t f_2(\sigma,\mathbb{B}(\sigma))\,d\mathbb{B}(\sigma),\ s<t\,.\end{aligned}$$

Proof. Assume that $f(t,x)$ has continuous partial derivatives of at least second order. Write $\mathbb{B}(t+dt)-\mathbb{B}(t)$ for the increment of $\mathbb{B}$ on $[t,t+dt]$. Using Taylor expansion we can write

$$\begin{aligned} &f(t+dt,\mathbb{B}(t+dt))-f(s,\mathbb{B}(s)) \\ &\quad= f_1(t,\mathbb{B}(t))\,dt+f_2(t,\mathbb{B}(t))\,d\mathbb{B}(t)\\ &\qquad+\frac{1}{2}\left[f_{11}(t,\mathbb{B}(t))+2f_{12}(t,\mathbb{B}(t))\,dt\,d\mathbb{B}(t)+f_{22}(t,\mathbb{B}(t))(d\mathbb{B}(t))^2\right]+\cdots\,.\end{aligned} \tag{3.27}$$

Now, let us introduce formally a multiplication table.

$$\begin{aligned} dt\,dt &= 0, & d\mathbb{B}(t)\,dt &= 0\,, \\ d\mathbb{B}(t)\,d\mathbb{B}(t) &= dt, & dt\,d\mathbb{B}(t) &= 0\,. \end{aligned}$$

As in classical calculus, higher-order terms in (3.27) are negligible, and so are the terms with factors $dt\,d\mathbb{B}(t)$ and $(dt)^2$. However, since we interpret $(d\mathbb{B}(t))^2$ as dt, the term with $(d\mathbb{B}(t))^2$ cannot be neglected. We then have

$$\begin{aligned} &f(t,\mathbb{B}(t)) - f(s,\mathbb{B}(s)) \\ &\quad = \int_s^t \left[f_1(\sigma,\mathbb{B}(\sigma)) + \frac{1}{2} f_{22}(\sigma,\mathbb{B}(\sigma)) \right] d\sigma + \int_s^t f_2(\sigma,\mathbb{B}(\sigma))\,d\mathbb{B}(\sigma),\; s<t\,, \end{aligned}$$

as desired.

Theorem 3.5. (Version II of Itô formula) *Let X be an Itô process given by*

$$dX(t) = a(t)\,dt + b(t)\,d\mathbb{B}(t) \tag{3.28}$$

with both, $a(t)$ and $b(t)$, being adapted to Brownian motion $\mathbb{B}$, and let $f(t,x)$ be a function whose second-order partial derivatives exist and are continuous. Then

$$\begin{aligned} &f(t,X(t)) - f(s,X(s)) \\ &\quad = \int_s^t \left[f_1(\sigma,X(\sigma)) + a(\sigma)\,f_2(\sigma,X(\sigma)) + \frac{1}{2}\,b(\sigma)^2 f_{22}(\sigma,X(\sigma)) \right] d\sigma \\ &\quad + \int_s^t b(\sigma)\,f_2(\sigma,X(\sigma))\,d\mathbb{B}(\sigma),\; s<t\,. \end{aligned} \tag{3.29}$$

Proof. To justify formula (3.29), we proceed as before. We use a Taylor expansion for $f(t+dt,X(t+dt)) - f(t,X(t))$ as in (3.27), where $\mathbb{B}$ is replaced with X, and X is defined in (3.28). Now, neglecting high-order terms, starting with terms involving $(dt)^2$ and $dt\,d\mathbb{B}(t)$, and making use of $(d\mathbb{B}(t))^2 = dt$, we obtain the desired formula.

Formula (3.29) is often written as follows:

$$\begin{aligned} &f(t,X(t)) - f(s,X(s)) \\ &\quad = \int_s^t \left[f_1(\sigma,X(\sigma)) + \frac{1}{2}\,b(\sigma)^2 f_{22}(\sigma,X(\sigma)) \right] d\sigma \\ &\quad + \int_s^t b(\sigma)\,f_2(\sigma,X(\sigma))\,dX(\sigma), \end{aligned}$$

where

$$dX(t) = a(t)\,dt + b(t)\,d\mathbb{B}(t)\,.$$

To illustrate the usefulness of Itô's formula, we provide some examples.

Example 3.16. Let us evaluate the stochastic integral $\int_0^t \mathbb{B}(s)\,d\mathbb{B}(s)$. To do this, take $f(t,x) = \frac{1}{2}x^2$. Noting $\mathbb{B}(0) = 0$ and applying Itô's formula yield

$$\frac{1}{2}\mathbb{B}(t)^2 = \int_0^t \frac{1}{2}\,ds + \int_0^t \mathbb{B}(s)\,d\mathbb{B}(s)$$
$$= \frac{1}{2}\,t + \int_0^t \mathbb{B}(s)\,d\mathbb{B}(s)\,.$$

Hence, $\int_0^t \mathbb{B}(s)\,d\mathbb{B}(s) = \frac{1}{2}\mathbb{B}(t)^2 - \frac{1}{2}\,t$.

Example 3.17. Consider the following stochastic equation:

$$dX(t) = rX(t) + cX(t)\,d\mathbb{B}(t)\,,$$

where $rX(t)\,dt$ represents exponential growth, $r > 0$, and where $cX(t)\,d\mathbb{B}(t)$ represents environmental variation, $c > 0$. Now, take $f(t,x) = \ln x$. Applying Itô's formula gives

$$\ln X(t) - \ln X(0) = \int_0^t \left(r - \frac{1}{2}\,c^2\right) ds + c\int_0^t d\mathbb{B}(s)$$
$$\ln\left(\frac{X(t)}{X(0)}\right) = \left(r - \frac{1}{2}\,c^2\right) t + c\mathbb{B}(t)\,.$$

Thus, $X(t) = X(0)\exp\left[\left(r - \frac{1}{2}\,c^2\right) t + c\mathbb{B}(t)\right]$.

3.3.5 Diffusion Process

To end this section, we now define infinitesimal drift and variance of a diffusion. Let us first introduce the following definition.

Definition 3.32. A stochastic process $X = \left\{X(t),\, t \geq 0\right\}$ is said to be a *Markov process* if

$$\mathbf{E}\Big[f(X(t+s))\,|\,\mathscr{F}_t\Big] = \mathbf{E}\Big[f(X(t+s))\,|\,X(t)\Big]$$

for all $t,\, s \geq 0$ and all $f : \mathbb{R} \to \mathbb{R}$ Borel measurable functions such that $\mathbf{E}\big|f(X(t))\big| < \infty$ for all t.

Under reasonable conditions on $\mu(\cdot)$ and $\sigma(\cdot)$, there exists a solution $X = \left\{X(t),\, t \geq 0\right\}$ to (3.18). The state stochastic process X is called a *diffusion* or *Itô process*. It is a Markov process with continuous paths and is time-homogeneous in the sense that

$$P_x\big\{X(t+h) \in \cdot\,|\,X(u) : 0 \leq u \leq t\big\} = P(h,\, X(t),\, \cdot)\,,$$

where

$$P(h,x,B) = P_x\big\{X(h) \in B\big\}\,.$$

Note that when $h > 0$ is small,

$$X(h)-X(0)=\int_0^h \mu(X(s))\,ds+\int_0^h \sigma(X(s))\,d\mathbb{B}(s)\approx\mu(X(0))h+\sigma(X(0))[\mathbb{B}(h)-\mathbb{B}(0)].$$

Hence,

$$E_x[X(h)-x]=\mu(x)h+o(h)$$

and

$$E_x\left[(X(h)-x)^2\right]=\sigma^2(x)h+o(h)$$

as $h \to 0$. As a result, $\mu(x)$ is called the *infinitesimal drift* of the diffusion X at x and $\sigma^2(x)$ is the *infinitesimal variance* of X at x.

Further discussions on this topic may be found in Doob [65] and Karlin and Taylor [108].

3.4 Wiener Process and Stochastic Integrals in a Hilbert Space

In the previous section, we presented the elements of Stochastic Calculus for real stochastic processes. These elements are also valid for stochastic processes taking their values in a separable Hilbert space. However, the extensions can be connected with some difficulties when we would be interested, for instance, in analytical properties of sample paths of such processes.

Of interest to us will be operator-valued random variables and their integrals. Let $\mathbb{K}$ and $\mathbb{H}$ be two separable Hilbert spaces with norms $\|\cdot\|_{\mathbb{K}}$, $\|\cdot\|_{\mathbb{H}}$ and inner products $\langle\cdot,\cdot\rangle_{\mathbb{K}}$, $\langle\cdot,\cdot\rangle_{\mathbb{H}}$, respectively. From now on, without further specification we always use the same symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces involved when no confusion is possible.

3.4.1 Wiener Process in a Separable Hilbert Space

Let $(\Omega,\mathscr{F},\mathbf{P},\mathscr{F}_t)$ be a filtered probability space and let $\beta_n(t)$ $(n=1,2,3,\ldots)$ be a sequence of real-valued standard Brownian motions mutually independent on this space.

Set

$$\mathbb{W}(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\,\beta_n(t)e_n,\ \ t\geq 0,$$

where $\lambda_n \geq 0$ $(n \geq 1)$ are nonnegative real numbers and $(e_n)_{n\geq 1}$ is the complete orthonormal basis in $\mathbb{K}$.

Let $\mathscr{Q} \in B(\mathbb{K},\mathbb{K})$ be the operator defined by $\mathscr{Q}e_n=\lambda_n e_n$ such that

$$\mathrm{Tr}\mathscr{Q} = \sum_{\mathrm{i}=1}^{\infty} \lambda_{\mathrm{i}} < \infty.$$

Clearly, $\mathbf{E}\mathbb{W}(t) = 0$ and for all $t \geq s \geq 0$, the distribution of $\mathbb{W}(t) - \mathbb{W}(s)$ is $\mathscr{N}(0,(t-s)\mathscr{Q})$.

The above-mentioned $\mathbb{K}$-valued stochastic process $\mathbb{W}(t)$ is called a $\mathscr{Q}$-*Wiener* process. In case the time set is $\mathbb{R}$, $\mathbb{W}$ can be obtained as follows: let $\{\mathbb{W}_i(t),\ t \in \mathbb{R}\},\ \ i = 1,2$, be independent $\mathbb{K}$-valued $\mathscr{Q}$-Wiener processes, then

$$\mathbb{W}(t) = \begin{cases} \mathbb{W}_1(t) & \text{if } t \geq 0, \\ \mathbb{W}_2(-t) & \text{if } t \leq 0, \end{cases}$$

is a $\mathscr{Q}$-Wiener process with $\mathbb{R}$ as time parameter and with values in $\mathbb{K}$.

3.4.2 Stochastic Integrals in a Hilbert Space

In order to define stochastic integrals with respect to the $\mathscr{Q}$-Wiener process $\mathbb{W}$, we introduce the subset $\mathbb{K}_0 = \mathscr{Q}^{\frac{1}{2}}\mathbb{K}$, which is a Hilbert space equipped with the norm

$$\left\|u\right\|_{\mathbb{K}_0} = \left\|\mathscr{Q}^{1/2}u\right\|_{\mathbb{K}}, \quad u \in \mathbb{K}_0\,,$$

and define a proper space of operators

$$\mathbb{L}_2^0 = \mathbb{L}_2^0(\mathbb{K}_0,\mathbb{H}) = \left\{\Psi \in B(\mathbb{K}_0,\mathbb{H}):\ \mathrm{Tr}\left[(\Psi\mathscr{Q}^{1/2})(\Psi\mathscr{Q}^{1/2})^*\right] < \infty\right\},$$

the space of all Hilbert–Schmidt operators from $\mathbb{K}_0$ into $\mathbb{H}$. It turns out that $\mathbb{L}_2^0$ is a separable Hilbert space with norm

$$\left\|\Psi\right\|_{\mathbb{L}_2^0}^2 = \mathrm{Tr}\left[(\Psi\mathscr{Q}^{1/2})(\Psi\mathscr{Q}^{1/2})^*\right] \quad \text{for any } \Psi \in \mathbb{L}_2^0\,.$$

Clearly, for any bounded linear operator $\Psi \in B(\mathbb{K},\mathbb{H})$, this norm reduces to

$$\left\|\Psi\right\|_{\mathbb{L}_2^0}^2 = \mathrm{Tr}\left[\Psi\mathscr{Q}\Psi^*\right].$$

For any $T \geq 0$, let $\Phi = \{\Phi(t),\, t \in [0,T]\}$, be an $\mathscr{F}_t$-adapted, $\mathbb{L}_2^0$-valued process, and for any $t \in [0,T]$, define the following norm:

$$\|\Phi\|_t := \left\{\mathbf{E}\int_0^t \mathrm{Tr}\left[(\Phi\mathscr{Q}^{1/2})(\Phi\mathscr{Q}^{1/2})^*\right] ds\right\}^{1/2}. \tag{3.30}$$

In general, we denote all $\mathbb{L}_2^0$-valued predictable processes Φ such that $\|\Phi\|_T < \infty$ by $\mathscr{U}^2([0,T],\mathbb{L}_2^0)$. The stochastic integral $\displaystyle\int_0^t \Phi(s)\,d\mathbb{W}(s) \in \mathbb{H}$ may be well defined for all $\Phi \in \mathscr{U}^2([0,T];\mathbb{L}_2^0)$ by

$$\int_0^t \Phi(s)\, d\mathbb{W}(s) = \mathrm{L}^2 - \lim_{n\to\infty} \sum_{i=0}^{n} \int_0^t \Phi(s)\, \sqrt{\lambda_i}\, e_i\, d\beta_i(s),\ t \in [0,T],$$

where $\mathbb{W}$ is the $\mathscr{Q}$-Wiener process defined above.

Proposition 3.22. *For arbitrary $T \geq 0$, let $\Phi \in \mathscr{U}^2([0,T];\mathbb{L}_2^0)$. Then the stochastic integral $\int_0^t \Phi(s)\, d\mathbb{W}(s)$ is a continuous, square integrable, $\mathbb{H}$-valued martingale on $[0,T]$ and*

$$\mathbf{E}\Big\| \int_0^t \Phi(t)\, d\mathbb{W}(s)\Big\|_{\mathbb{H}}^2 = \|\Phi\|_t^2,\ t \in [0,T]. \tag{3.31}$$

In fact, the stochastic integral $\int_0^t \Phi(s)\, d\mathbb{W}(s),\ t \geq 0$ may be extended to any $\mathbb{L}_2^0$-valued adapted process Φ satisfying

$$\mathbf{P}\Big\{\omega : \int_0^t \|\Phi(\omega,s)\|_{\mathbb{L}_2^0}^2\, ds < \infty,\ 0 \leq t \leq T\Big\} = 1.$$

Moreover, we may deduce the following generalized relation of (3.31):

$$\mathbf{E}\Big\| \int_0^t \Phi(t)\, d\mathbb{W}(s)\Big\|_{\mathbb{H}}^2 \leq \mathbf{E}\int_0^t \|\Phi(s)\|_{\mathbb{L}_2^0}^2\, ds,\ 0 \leq t \leq T. \tag{3.32}$$

Note that the equality holds if the right-hand side of this inequality is finite.

The following proposition is a particular case of the Burkholder–Davis–Gundy inequality.

Proposition 3.23. *For any $p \geq 2$ and for arbitrary $\mathbb{L}_2^0$-valued predictable process $\Phi(t)$, $t \in [0,T]$, one has*

$$\mathbf{E}\Big[\sup_{s\in[0,t]} \Big\| \int_0^s \Phi(s)\, d\mathbb{W}(s)\Big\|^p\Big] \leq C_p \mathbf{E}\Big[\int_0^t \|\Phi(s)\|_{\mathbb{L}_2^0}^2\, ds\Big]^{p/2} \tag{3.33}$$

for some constant $C_p > 0$.

For a proof, see, e.g., Da Prato and Zabczyk [47] or Seidler and Sobukawa [164].

Finally, let us quote Theorem 3 from Da Prato and Zabczyk [46], which is a stochastic version of the Fubini theorem and enables us to interchange stochastic and Bochner integrals.

Proposition 3.24. *Let $(G,\mathscr{G},\mu)$ be a measure space, let $h : \Omega \times [0,T] \times G \to \mathbb{L}_2^0$ be an $\mathscr{F} \otimes \mathscr{B}([0,T]) \otimes \mathscr{G}$-measurable mapping such that $h(\cdot,\cdot,x)$ is an $(\mathscr{F}_t)$-adapted stochastic process for each $x \in G$ and*

$$\int_G \Big(\int_0^T \mathbf{E}\|h(t,x)\|_{\mathbb{L}_2^0}^2\Big)^{1/2} d\mu(x) < \infty.$$

Then

$$\int_G \Big(\int_0^T h(t,x)\, d\mathbb{W}(t) \Big)\, d\mu(x) = \int_0^T \Big(\int_G h(t,x)\, d\mu(x) \Big)\, d\mathbb{W}(t) \ \ \mathbf{P}\ a.s.$$

3.4.3 Stochastic Convolution Integrals

Let $(\Omega, \mathscr{F}, \mathbf{P}, \mathscr{F}_t)$ be a filtered probability space. Let $\Delta = \{(s,t) : 0 \le s < t \le T\}$ and suppose that $U = \{U(t,s) : (s,t) \in \Delta\}$ is an evolution operator as in Chapter 2. Denote by $\mathscr{M}$ the σ-field of $(\mathscr{F}_t)$-progressively measurable sets over $\Omega \times \mathbb{R}_+$ and by Ψ an $\mathscr{M}$-measurable $\mathbb{L}_2^0$-valued process.

Define

$$I(t) = \int_0^t U(t,s)\Psi(s)\, d\mathbb{W}(s),\ 0 \le t \le T\,.$$

This integral is well defined provided that

$$\int_0^t \big\| U(t,s)\Psi(s) \big\|_{\mathbb{L}_2^0}^2\, ds < \infty\ \mathbf{P}\ \text{a.s.},\ 0 \le t \le T\,.$$

Such an integral is called a *stochastic convolution* integral.

From the formula

$$\int_\sigma^t (t-s)^{\xi-1}(s-\sigma)^{-\xi}\, ds = \frac{\pi}{\sin(\pi\xi)},\ \text{for } \sigma \le s \le t,\ \xi \in (0,1)\,,$$

established in Da Prato, Kawapian, and Zabczyk [45], it follows that

$$\int_0^t U(t,s)\Psi(s)\, d\mathbb{W}(s) = \frac{\sin \pi\xi}{\pi}\,(R_\xi \mathbb{S}_\Psi)(t)\ \text{a.s.}, \tag{3.34}$$

where

$$(R_\xi \mathbb{S}_\Psi)(t) = \int_0^t (t-s)^{\xi-1} U(t,s) Z(s)\, ds$$

with

$$\mathbb{S}_\Psi(s) = \int_0^s (s-\sigma)^{-\xi} U(s,\sigma)\Psi(\sigma)\, d\mathbb{W}(\sigma)\,.$$

The use of the representation (3.34) is the very core of the factorization method as treated in Da Prato, Kawapian, and Zabczyk [45] and Da Prato and Zabczyk [48]. It is possible to derive estimates for $I(t)$ provided that the evolution operator U is exponentially stable, that is,

$$\big\| U(t,s) \big\| \le M e^{-\delta(t-s)} \tag{3.35}$$

for some constants $M > 0$ and $\delta > 0$ and for all $t \ge s \ge 0$.

Let $\mathbb{H}_\alpha$, $\alpha \in [0,1]$ be intermediate Banach spaces such that $\mathbb{H}_0 = \mathbb{H}$, $\mathbb{H}_\beta$ is continuously embedded into $\mathbb{H}_\nu$ whenever $1 \ge \beta \ge \nu \ge 0$, and for each $\rho \in [0,1]$ there exists a constant L_ρ such that

$$U(t,s) \in B(\mathbb{H}, \mathbb{H}_\rho) \text{ and } \big\|U(t,s)\big\| \leq \frac{L_\rho}{(t-s)^\rho}$$

for all $0 \leq s < t \leq T$.

As in Chapter 2, we shall denote the norm in $\mathbb{H}_\alpha$ simply by $\big\|\cdot\big\|_\alpha$.

We now state the maximal inequality.

Proposition 3.25. *Let $p > 2$, $\alpha \in [0, \frac{p-2}{2p})$. Let $\Psi : \Omega \times [0,T] \to \mathbb{L}_2^0$ be an $(\mathscr{F}_t)$-adapted measurable stochastic process such that*

$$\int_0^T \mathbf{E}\big\|\Psi(s)\big\|_{\mathbb{L}_2^0}^p \, ds < \infty.$$

Then

$$\mathbf{E}\left[\sup_{0\leq t\leq T} \left\|\int_0^t U(t,s)\Psi(s)\, d\mathbb{W}(s)\right\|_\alpha^p\right] \leq C \int_0^T \mathbf{E}\big\|\Psi(s)\big\|_{\mathbb{L}_2^0}^p \, ds,$$

where the constant C depends only on p, α, T, and U.

For the proof, we refer the reader to Seidler [163].

The above proposition can be extended to $(\mathscr{F}_t)$-adapted measurable stochastic processes whose time set is $\mathbb{R}$.

Proposition 3.26. *Let $p > 2$, $0 < \alpha < 1$, $\alpha + \frac{1}{p} < \xi < \frac{1}{2}$, and $\Psi : \Omega \times \mathbb{R} \to \mathbb{L}_2^0$ be an $(\mathscr{F}_t)$-adapted measurable stochastic process such that*

$$\sup_{t\in\mathbb{R}} \mathbf{E}\big\|\Psi(t)\big\|_{\mathbb{L}_2^0}^p < \infty.$$

Then

(i) $$\mathbf{E}\left\|\int_{-\infty}^t (t-s)^{-\xi} U(t,s)P(s)\Psi(s)\, d\mathbb{W}(s)\right\|^p \leq C_p N^p C_1(\Gamma, \xi, \delta, p) \sup_{t\in\mathbb{R}} \mathbf{E}\big\|\Psi(t)\big\|_{\mathbb{L}_2^0}^p;$$

(ii) $$\mathbf{E}\left\|\int_{-\infty}^t U(t,s)P(s)\Psi(s)\, d\mathbb{W}(s)\right\|_\alpha^p \leq C_p M(\alpha)^p C_2(\Gamma, \alpha, \xi, \delta, p)\, C_1(\Gamma, \xi, \delta, p) \sup_{t\in\mathbb{R}} \mathbf{E}\big\|\Psi(t)\big\|_{\mathbb{L}_2^0}^p;$$

(iii) $$\mathbf{E}\left\|\int_t^\infty U(t,s)Q(s)\Psi(s)\, d\mathbb{W}(s)\right\|_\alpha^p \leq C_p M(\alpha)^p C_3(\Gamma, \alpha, \xi, \delta, p)\, C_1(\Gamma, \xi, \delta, p) \sup_{t\in\mathbb{R}} \mathbf{E}\big\|\Psi(t)\big\|_{\mathbb{L}_2^0}^p$$

where

$$C_1(\Gamma,\xi,\delta,p) = N^p\left[\Gamma\big(1-\frac{2p\xi}{p-2}\big)\right]^{\frac{p-2}{2}}(2\delta)^{\frac{4\xi-1}{2}p},$$

$$C_2(\Gamma,\alpha,\xi,\delta,p) = \left|\frac{\sin(\pi\xi)}{\pi}\right|^p\left(\frac{1}{\delta}\right)\left[\Gamma\Big(1-\frac{p}{p-1}(1+\alpha-\xi)\Big)\right]^{p-1}\delta^{p(\alpha-\xi)},$$

$$C_3(\Gamma,\xi,\delta,p) = \left|\frac{\sin(\pi\xi)}{\pi}\right|^p\left(\frac{1}{\delta}\right)\left[\Gamma\Big(1-\frac{p}{p-1}(1-\xi)\Big)\right]^{p-1}\delta^{-p\xi},$$

with Γ a classical Gaussian function. Here, $P(t)$, $t\in\mathbb{R}$ are projections that are uniformly bounded and strongly continuous in t, $Q(t) = I-P(t)$, and $\mathbb{W}$ is a $\mathscr{Q}$-Wiener with values in $\mathbb{K}$ and with time set $\mathbb{R}$.

Proof. (i) A direct application of Proposition 3.23 and Hölder's inequality with the help of (3.35) allows us to write

$$\begin{aligned}
&\mathbf{E}\left\|\int_{-\infty}^t (t-\sigma)^{-\xi}U(t,\sigma)P(\sigma)\Psi(\sigma)\,d\mathbb{W}(\sigma)\right\|^p\\
&\leq C_p\mathbf{E}\left[\int_{-\infty}^t (t-\sigma)^{-2\xi}\left\|U(t,\sigma)P(\sigma)\Psi(\sigma)\right\|^2_{\mathbb{L}_2^0}d\sigma\right]^{p/2}\\
&\leq C_pN^p\mathbf{E}\Big[\int_{-\infty}^t (t-\sigma)^{-2\xi}e^{-2\delta(t-\sigma)}\left\|\Psi(\sigma)\right\|^2_{\mathbb{L}_2^0}d\sigma\Big]^{p/2}\\
&\leq C_pN^p\Big(\int_{-\infty}^t (t-\sigma)^{-2\xi}e^{-2\delta(t-\sigma)}\,d\sigma\Big)^{p-1}\Big(\int_{-\infty}^t e^{-2\delta(t-\sigma)}\mathbf{E}\left\|\Psi(\sigma)\right\|^p_{\mathbb{L}_2^0}d\sigma\Big)\\
&\leq C_pN^p\Big(\Gamma(1-\frac{2p\xi}{p-2})(2\delta)^{\frac{2p\xi}{p-2}-1}\Big)^{\frac{p-2}{2}}\Big(\frac{1}{2\delta}\Big)\sup_{t\in\mathbb{R}}\mathbf{E}\left\|\Psi(t)\right\|^p_{\mathbb{L}_2^0}\\
&\leq C_pC_1(\Gamma,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\left\|\Psi(t)\right\|^p_{\mathbb{L}_2^0}.
\end{aligned}$$

To prove (ii), we use the factorization method of the stochastic convolution integral:

$$\int_{-\infty}^t U(t,s)P(s)\Psi(s)\,d\mathbb{W}(s) = \frac{\sin\pi\xi}{\pi}\,(R_\xi\mathbb{S}_\Psi)(t)\ \text{a.s.}$$

where

$$(R_\xi\mathbb{S}_\Psi)(t) = \int_{-\infty}^t (t-s)^{\xi-1}U(t,s)P(s)\mathbb{S}_\Psi(s)\,ds$$

with

$$\mathbb{S}_\Psi(s) = \int_{-\infty}^s (s-\sigma)^{-\xi}U(s,\sigma)P(\sigma)\Psi(\sigma)\,d\mathbb{W}(\sigma),$$

and ξ satisfying $\alpha+\frac{1}{p}<\xi<\frac{1}{2}$.

We can now evaluate $\mathbf{E}\left\|\int_{-\infty}^t U(t,s)P(s)\Psi(s)\,d\mathbb{W}(s)\right\|^p_\alpha$:

$$
\begin{aligned}
&\mathbf{E}\left\|\int_{-\infty}^{t} U(t,s)P(s)\Psi(s)\,d\mathbb{W}(s)\right\|_{\alpha}^{p}\\
&\leq \left|\frac{\sin(\pi\xi)}{\pi}\right|^{p}\mathbf{E}\Big[\int_{-\infty}^{t}(t-s)^{-\xi}\big\|U(t,s)P(s)\mathbb{S}_{\Psi}(s)\big\|_{\alpha}\,ds\Big]^{p}\\
&\leq M(\alpha)^{p}\left|\frac{\sin(\pi\xi)}{\pi}\right|^{p}\mathbf{E}\Big[\int_{-\infty}^{t}(t-s)^{\xi-\alpha-1}e^{-\delta(t-s)}\big\|\mathbb{S}_{\Psi}(s)\big\|_{\alpha}\,ds\Big]^{p}\\
&\leq M(\alpha)^{p}\left|\frac{\sin(\pi\xi)}{\pi}\right|^{p}\Big(\int_{-\infty}^{t}(t-s)^{\frac{p}{p-1}(\xi-\alpha-1)}e^{-\delta(t-s)}\,ds\Big)^{p-1}\times\\
&\times\Big(\int_{-\infty}^{t}e^{-\delta(t-s)}\mathbf{E}\big\|\mathbb{S}_{\Psi}(s)\big\|^{p}\,ds\Big)\\
&\leq M(\alpha)^{p}C_{2}(\Gamma,\alpha,\xi,\delta,p)\sup_{s\in\mathbb{R}}\mathbf{E}\big\|\mathbb{S}_{\Psi}(s)\big\|^{p}.
\end{aligned}
\tag{3.36}
$$

On the other hand, it follows from part (i) that

$$
\mathbf{E}\big\|\mathbb{S}_{\Psi}(t)\big\|^{p}\leq C_{p}\,C_{1}(\Gamma,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi(t)\big\|_{\mathbb{L}_{2}^{0}}^{p}.
\tag{3.37}
$$

Thus,

$$
\begin{aligned}
&\mathbf{E}\big\|\int_{-\infty}^{t} U(t,s)P(s)\Psi(s)\,d\mathbb{W}(s)\big\|_{\alpha}^{p}\\
&\leq C_{p}\,M(\alpha)^{p}C_{2}(\Gamma,\alpha,\xi,\delta,p)\,C_{1}(\Gamma,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi(t)\big\|_{\mathbb{L}_{2}^{0}}^{p}.
\end{aligned}
$$

To prove (iii), we also use the factorization method:

$$
\begin{aligned}
&\mathbf{E}\left\|\int_{t}^{\infty} U(t,s)Q(s)\Psi(s)\,d\mathbb{W}(s)\right\|_{\alpha}^{p}\\
&\leq \left|\frac{\sin(\pi\xi)}{\pi}\right|^{p}\mathbf{E}\Big[\int_{t}^{\infty}(s-t)^{-\xi}\big\|U(t,s)Q(s)\mathbb{S}_{\Psi}(s)\big\|_{\alpha}\,ds\Big]^{p}\\
&\leq M(\alpha)^{p}\left|\frac{\sin(\pi\xi)}{\pi}\right|^{p}\mathbf{E}\Big[\int_{t}^{\infty}(s-t)^{\xi-1}e^{-\frac{\delta}{2}(t-s)}\big\|\mathbb{S}_{\Psi}(s)\big\|\,ds\Big]^{p}\\
&\leq M(\alpha)^{p}\left|\frac{\sin(\pi\xi)}{\pi}\right|^{p}\Big(\int_{t}^{\infty}(s-t)^{\frac{p}{p-1}(\xi-1)}e^{-\frac{\delta}{2}(s-t)}\,ds\Big)^{p-1}\times\\
&\times\Big(\int_{t}^{\infty}e^{-\frac{\delta}{2}(s-t)}\mathbf{E}\big\|\mathbb{S}_{\Psi}(s)\big\|^{p}\,ds\Big)\\
&\leq M(\alpha)^{p}C_{3}(\Gamma,\alpha,\xi,\delta,p)\sup_{s\in\mathbb{R}}\mathbf{E}\big\|\mathbb{S}_{\Psi}(s)\big\|^{p}.
\end{aligned}
\tag{3.38}
$$

It follows from (3.37) that

$$
\begin{aligned}
&\mathbf{E}\left\|\int_{t}^{\infty} U(t,s)Q(s)\Psi(s)\,d\mathbb{W}(s)\right\|_{\alpha}^{p}\\
&\leq C_{p}\,M(\alpha)^{p}C_{3}(\Gamma,\alpha,\xi,\delta,p)\,C_{1}(\Gamma,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi(t)\big\|_{\mathbb{L}_{2}^{0}}^{p}.
\end{aligned}
$$

3.5 Existence of Solutions of Stochastic Differential Equations in a Hilbert Space

During the last few decades, stochastic differential equations in a separable Hilbert space have been of great interest to several mathematicians and various results on the existence, uniqueness, stability, and other quantitative and qualitative properties of solutions have been established. For example, in their book [47], Da Prato and Zabczyk established systematic theory of the existence and uniqueness and ergodicity theory for infinite-dimensional systems. The literature relative to those equations is quite extensive; for more on this topic and related applications we refer the reader to Appleby [9], Caraballo and Kai Liu [33], Kai Liu [106], and Luo [133, 134].

3.5.1 Existence and Uniqueness

Let us first consider the following stochastic differential equation on $\mathbb{H}$ of the form

$$\begin{cases} dX(t) = f(X(t))\,dt + g(X(t))\,d\mathbb{W}(t), \quad t \in [0,T] \\ X(0) = x_0, \end{cases} \tag{3.39}$$

where $f : \mathbb{H} \to \mathbb{H}$ and $g : \mathbb{H} \to \mathbb{L}_2^0$ are Borel measurable, $x_0 \in \mathbb{H}$ is either nonrandom or $\mathscr{F}$-measurable, and $\mathbb{W}$ is a $\mathscr{Q}$-Wiener process on $\mathbb{H}$. This can be written as a stochastic integral equation

$$X(t) = x_0 + \int_0^t f(X(s))\,ds + \int_0^t g(X(s))\,d\mathbb{W}(s)\,.$$

Moreover, we assume that f and g satisfy the following Lipschitz condition:
(A) There exists a positive constant C such that

$$\|f(x) - f(y)\|_{\mathbb{H}} \le C\,\|x - y\|_{\mathbb{H}}$$

and

$$\|g(x) - g(y)\|_{\mathbb{L}_2^0} \le C\,\|x - y\|_{\mathbb{H}}\,.$$

Therefore, according to Yor [185] and Miyahara [144], we have

Proposition 3.27. *There exists a unique solution X of Eq. (3.39), which is a diffusion with generator L:*

$$Lh(x) = \langle h'(x), f(x)\rangle_{\mathbb{H}} + (1/2)\mathrm{Tr}\Big[\mathrm{g}^*(\mathrm{x})\mathrm{h}''(\mathrm{x})\mathrm{g}(\mathrm{x})\Big]\,.$$

Moreover, X has continuous paths, i.e.,

$$\mathbf{P}\Big\{\omega : \lim_{t\to s} \|X(\omega,t) - X(\omega,s)\|_{\mathbb{H}} = 0\Big\} = 1\,.$$

Proof. See Yor [185].

3.5.2 L^2-Bounded Solutions

Here, we are interested in studying L^2-boundedness of the solution of the following stochastic differential equations of the form

$$dX(t) = \mathscr{A}X(t) + f(t,X(t))\,dt + g(t,X(t))\,d\mathbb{W}(t),\ t \in \mathbb{R} \tag{3.40}$$

with the initial condition

$$X(0) = x_0 \in D(\mathscr{A}),$$

where $\mathscr{A} : \mathscr{D} = D(\mathscr{A}) \subset \mathbb{H} \mapsto \mathbb{H}$ is a densely defined closed (possibly unbounded) linear operator, $f : \mathbb{R}\times\mathbb{H} \to \mathbb{H}$ and $g : \mathbb{R}\times\mathbb{H} \to \mathbb{L}_2^0$ are jointly continuous functions, and $\mathbb{W}(t)$ is a $\mathscr{Q}$-Wiener process with values in $\mathbb{K}$.

In what follows we adopt the following assumptions:

$(3H)_0$ The operator A is the infinitesimal generator of a uniformly exponentially stable semigroup $(T(t))_{t\geq 0}$ defined on $\mathbb{H}$, that is, there exist constants $M, \delta > 0$ such that

$$\|\,T(t)\| \leq Me^{-\delta t}, \quad t \geq 0.$$

$(3H)_1$ The coefficients $f(\cdot,\,\cdot)$ and $g(\cdot,\,\cdot)$ satisfy the following Lipschitz and linear growth conditions: there exist positive constants C_i, $i = 1,2$ such that the following conditions are satisfied:

$$\begin{aligned}\|f(t,x) - f(t,y)\| &\leq C_1\,\|x-y\|,\\ \|g(t,x) - g(t,y)\|_{\mathbb{L}_2^0} &\leq C_2\,\|x-y\|,\end{aligned}$$

for all $t \in \mathbb{R}$, x, and $y \in \mathbb{H}$.

$(3H)_2$ There exists a constant C such that

$$\|f(t,x)\|^2 + \|g(t,x)\|^2_{\mathbb{L}_2^0} \leq C,$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{H}$.

For convenience, we recall from Ichikawa [104] two kinds of solutions of (3.40).

Definition 3.33. A stochastic process $\{X(t),\, t \in \mathbb{R}\}$ is said to be a strong solution to (3.40) if

(i) $X(t)$ is adapted to $\mathscr{F}_t$;
(ii) $X(t)$ is continuous in t almost surely;
(iii) $X(t) \in \mathscr{D}$ for any $t \geq 0$, $\displaystyle\int_{\infty}^{T} \|AX(t)\|\,dt < \infty$ almost surely for any $T > 0$, and

$$X(t) = X(s) + \int_s^t AX(\sigma)\,d\sigma + \int_s^t f(\sigma, X(\sigma))\,d\sigma + \int_0^t g(\sigma, X(\sigma))\,d\mathbb{W}(\sigma)$$

for all $t \geq s$ with probability one.

In most situations, we find that the concept of strong solution is too limited to include important examples. There is a weaker concept, mild solution, which is found to be more appropriate for practical purposes.

Definition 3.34. A stochastic process $\{X(t), t \in \mathbb{R}\}$ is said to be a mild solution to (3.40) if

(i) $X(t)$ is adapted to $\mathscr{F}_t$;
(ii) $X(t)$ is continuous in t almost surely;
(iii) X is measurable with $\int_{-\infty}^T \|X(t)\|^2\,dt < \infty$ almost surely for any $T > 0$, and

$$X(t) = T(t)X(s) + \int_s^t T(t-\sigma)f(\sigma, X(\sigma))\,d\sigma + \int_s^t T(t-\sigma)g(\sigma, X(\sigma))\,d\mathbb{W}(\sigma)$$

for all $t \geq s$ with probability one.

Let $p \geq 2$ and denote by $L^p(\Omega, \mathbb{H})$ the collection of all strongly measurable, p-th integrable $\mathbb{H}$-valued random variables. It is then routine to check that $L^p(\Omega, \mathbb{H})$ is a Banach space when it is equipped with its norm defined by

$$\|V\|_{L^p(\Omega,\mathbb{H})} := \left[\mathbf{E}\,\|V\|^p\right]^{1/p},$$

for each $V \in L^p(\Omega, \mathbb{H})$.

Let $BUC(\mathbb{R}; L^p(\Omega; \mathbb{H}))$ stand for the collection of all processes $X = \Big\{X(t), t \in \mathbb{R}\Big\}$, which are bounded and uniformly continuous in $L^p(\Omega, \mathbb{H})$. We can, and do, speak of such a process as a function X, which goes from $\mathbb{R}$ into $L^p(\Omega; \mathbb{H})$. It is then easy to check that $BUC(\mathbb{R}; L^p(\Omega; \mathbb{H}))$ is a Banach space when it is equipped with a norm defined by

$$\|X\|_\infty = \sup_{t \in \mathbb{R}} \|X(t)\|_{L^p(\Omega,\mathbb{H})}.$$

In this section for simplicity we assume that $p = 2$.

We have the following well-known theorem.

Theorem 3.6. *Suppose that the assumptions* $(3H)_0$, $(3H)_1$, and $(3H)_2$ *hold. Equation (3.40) has a unique uniformly continuous and* L^2*-bounded mild solution* $X(t)$, *which can be explicitly expressed as follows:*

$$X(t) = \int_{-\infty}^t T(t-\sigma)f(\sigma, X(\sigma))\,d\sigma + \int_{-\infty}^t T(t-\sigma)g(\sigma, X(\sigma))\,d\mathbb{W}(\sigma)$$

for each $t \in \mathbb{R}$ *whenever* $\Theta := 2M^2\left(\dfrac{C_1}{\delta^2} + \dfrac{C_2}{\delta}\right) < 1$.

Proof. Define an operator Φ on $BUC(\mathbb{R}; L^2(\Omega;\mathbb{H}))$ as follows:

$$\Phi X(t) = \int_{-\infty}^{t} T(t-\sigma)f(\sigma, X(\sigma))\,d\sigma + \int_{-\infty}^{t} T(t-\sigma)g(\sigma, X(\sigma))\,d\mathbb{W}(\sigma)\,.$$

Let us first show that $\Phi X(\cdot)$ is uniformly continuous whenever X is. Clearly, the mappings $\sigma \to f(\sigma, X(\sigma))$ and $\sigma \to g(\sigma, X(\sigma))$ are continuous and uniformly L^2-bounded. That is, for any $\varepsilon > 0$, there is an $h > 0$ sufficiently small such that

$$\mathbf{E}\big\|f(t+h, X(t+h)) - f(t, X(t))\big\|^2 < \frac{\delta^2}{4M^2}\,\varepsilon$$

and

$$\mathbf{E}\big\|g(t+h, X(t+h)) - g(t, X(t))\big\|^2_{\mathbb{L}_2^0} < \frac{\delta}{2M^2}\varepsilon\,,$$

for all $t \in \mathbb{R}$.
Then

$$\begin{aligned}
&\mathbf{E}\big\|\Phi X(t+h) - \Phi X(t)\big\|^2 \\
&\quad \le 2\mathbf{E}\left\|\int_{-\infty}^{t} T(t-\sigma)\big[f(\sigma+h, X(\sigma+h)) - f(\sigma, X(\sigma))\big]\,d\sigma\right\|^2 \\
&\qquad +2\left\|\int_{-\infty}^{t} T(t-\sigma)\big[g(\sigma+h, X(\sigma+h)) - g(\sigma, X(\sigma))\big]\,d\mathbb{W}(\sigma)\right\|^2.
\end{aligned}$$

Using assumption $(3H)_0$, the Cauchy–Schwarz inequality, and isometry identity, we have

$$
\begin{aligned}
&\mathbf{E}\big\|\Phi X(t+h)-\Phi X(t)\big\|^2\\
&\le 2\mathbf{E}\Big[\int_{-\infty}^{t}\big\|T(t-\sigma)\big\|\big\|f(\sigma+h,X(\sigma+h))-f(\sigma,X(\sigma))\big\|\,d\sigma\Big]^2\\
&\quad+2\int_{-\infty}^{t}\big\|T(t-\sigma)\big\|^2\,\mathbf{E}\big\|g(\sigma+h,X(\sigma+h))-g(\sigma,X(\sigma))\big\|^2\,d\sigma\\
&\le 2M^2\,\mathbf{E}\Big(\int_{-\infty}^{t}e^{-\delta(t-\sigma)}d\sigma\Big)\Big(\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\big\|f(\sigma+h,X(\sigma+h))-f(\sigma,X(\sigma))\big\|^2\,d\sigma\Big)\\
&\quad+2M^2\int_{-\infty}^{t}e^{-2\delta(t-\sigma)}\,\mathbf{E}\big\|g(\sigma+h,X(\sigma+h))-g(\sigma,X(\sigma))\big\|^2\,d\sigma\\
&\le 2M^2\Big(\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\,d\sigma\Big)^2\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|f(\sigma+h,X(\sigma+h))-f(\sigma,X(\sigma))\big\|^2\Big)\\
&\quad+2M^2\Big(\int_{-\infty}^{t}e^{-2\delta(t-\sigma)}\,d\sigma\Big)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|g(\sigma+h,X(\sigma+h))-g(\sigma,X(\sigma))\big\|^2\\
&\le 2\,\frac{M^2}{\delta^2}\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|f(\sigma+h,X(\sigma+h))-f(\sigma,X(\sigma))\big\|^2\\
&\quad+\frac{M^2}{\delta}\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|g(\sigma+h,X(\sigma+h))-g(\sigma,X(\sigma))\big\|^2\\
&\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

Next, we show that $\Phi X(\cdot)$ is L^2-bounded. For a fixed $t\in\mathbb{R}$, we have

$$
\begin{aligned}
\mathbf{E}\big\|\Phi X(t)\big\|^2 &\le 2\,\mathbf{E}\Big\|\int_{-\infty}^{t}T(t-s)f(s,X(s))\,ds\Big\|^2\\
&\quad+2\,\mathbf{E}\Big\|\int_{-\infty}^{t}T(t-s)g(s,X(s))\,d\mathbb{W}(s)\Big\|^2\\
&=\mathrm{I}_1+\mathrm{I}_2\,.
\end{aligned}
$$

Using assumption $(3H)_0$, an application of the Cauchy–Schwarz inequality, followed by $(3H)_2$, gives us

$$
\begin{aligned}
\mathrm{I}_1 &\le 2M^2\,\mathbf{E}\Big[\Big(\int_{-\infty}^{t}e^{-\delta(t-s)}\big\|f(s,X(s))\big\|\Big)^2\Big]\\
&\le 2M^2\Big(\int_{-\infty}^{t}e^{-\delta(t-s)}\,ds\Big)\Big(\int_{0}^{t}e^{-\delta(t-s)}\mathbf{E}\big\|f(s,X(s))\big\|^2\,ds\Big)\\
&\le 2\Big(\int_{-\infty}^{t}e^{-\delta(t-s)}\,ds\Big)^2\sup_{s\ge 0}\mathbf{E}\big\|f(s,X(s))\big\|^2\\
&\le 2C\cdot\frac{M^2}{\delta^2}\,.
\end{aligned}
$$

As to I_2, in a similar manner (with the additional help of Itô isometry), we have

$$
\begin{aligned}
\mathrm{I}_2 &\leq 2 \int_{-\infty}^{t} \left\| T(t-s) \right\|^2 \mathbf{E} \left\| g(s,X_s) \right\|_{\mathbb{L}_2^0}^2 ds \\
&\leq 2 \cdot M^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} \mathbf{E} \left\| g(s,X_s) \right\|_{\mathbb{L}_2^0}^2 ds \\
&\leq 2 \cdot M^2 \Big(\int_{-\infty}^{t} e^{-2\delta(t-s)} \, ds \Big) \sup_{s \in \mathbb{R}} \mathbf{E} \left\| g(s,X(s)) \right\|_{\mathbb{L}_2^0}^2 \\
&\leq C \cdot \frac{M^2}{\delta} .
\end{aligned}
$$

Combining, we conclude that

$$
\mathbf{E} \left\| \Phi X(t) \right\|^2 \leq 2M^2 \cdot C \Big(\frac{1}{\delta^2} + \frac{1}{\delta} \Big) \tag{3.41}
$$

for all $t \in \mathbb{R}$.

Finally, we will show that Φ is a contraction. Let X and Y in $BUC(\mathbb{R}; L^2(\Omega, \mathbb{H}))$. Proceeding as before, we obtain

$$
\begin{aligned}
&\mathbf{E} \left\| \Phi X(t) - \Phi Y(t) \right\|^2 \\
&\leq 2\mathbf{E} \left\| \int_{-\infty}^{t} T(t-\sigma) \left[f(\sigma, X(\sigma)) - f(\sigma, Y(\sigma)) \right] d\sigma \right\|^2 \\
&\quad + 2\mathbf{E} \left\| \int_{-\infty}^{t} T(t-\sigma) \left[g(\sigma, X(\sigma)) - g(\sigma, Y(\sigma)) \right] d\mathbb{W}(\sigma) \right\|^2 .
\end{aligned}
$$

Using assumption $(3H)_0$, an application of the Cauchy–Schwarz inequality, isometry identity, followed by $(3H)_2$, gives

$$
\begin{aligned}
&\mathbf{E} \left\| \Phi X(t) - \Phi Y(t) \right\|^2 \\
&\leq 2M^2 \Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \, d\sigma \Big) \Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \, \mathbf{E} \left\| f(\sigma, X(\sigma)) - f(\sigma, Y(\sigma)) \right\|^2 d\sigma \Big) \\
&\quad + 2M^2 \int_{-\infty}^{t} e^{-2\delta(t-\sigma)} \mathbf{E} \left\| g(\sigma, X(\sigma)) - g(\sigma, Y(\sigma)) \right\|^2 d\sigma \\
&\leq 2M^2 \cdot C_1 \Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \, d\sigma \Big) \Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \, \mathbf{E} \left\| X(\sigma) - Y(\sigma) \right\|^2 d\sigma \Big) \\
&\quad + 2M^2 \cdot C_2 \int_{-\infty}^{t} e^{-2\delta(t-\sigma)} \mathbf{E} \left\| X(\sigma) - Y(\sigma) \right\|^2 d\sigma \\
&\leq 2M^2 \cdot C_1 \Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \, d\sigma \Big)^2 \sup_{\sigma \in \mathbb{R}} \mathbf{E} \left\| X(\sigma) - Y(\sigma) \right\|^2 \\
&\quad + 2M^2 \cdot C_2 \Big(\int_{-\infty}^{t} e^{-2\delta(t-\sigma)} \, d\sigma \Big) \sup_{\sigma \in \mathbb{R}} \mathbf{E} \left\| X(\sigma) - Y(\sigma) \right\|^2 \\
&\leq 2M^2 \Big(\frac{C_1}{\delta^2} + \frac{C_2}{\delta} \Big) \left\| X - Y \right\|_\infty^2 \\
&\leq \Theta \cdot \left\| X - Y \right\|_\infty^2 .
\end{aligned}
$$

Consequently, if $\Theta < 1$, then Φ is a contraction mapping and this completes the proof.

3.5.3 Stochastic Delay Differential Equation and Exponential Stability

Let us now allow the coefficients of the stochastic differential equation (3.40) to depend on values in the past. We then obtain the so-called stochastic delay differential equation.

Here, we are interested in studying the following stochastic delay differential equations of the form

$$dX(t) = \mathscr{A}X(t) + f(t,X_t)\,dt + g(t,X_t)\,d\mathbb{W}(t),\ t \in \mathbb{R}_+ \tag{3.42}$$

with the initial condition

$$X(\cdot) = \varphi(\cdot) \in C([-\tau,0],\mathbb{H}),$$

where $\mathscr{A} : \mathscr{D} = D(\mathscr{A}) \subset \mathbb{H} \to \mathbb{H}$ is a densely defined closed (possibly unbounded) linear operator, the history $X_t \in C_\tau \equiv C([-\tau,0],\mathbb{H})$ with $\tau > 0$ (X_t being defined by $X_t(\theta) := X(t+\theta)$ for each $\theta \in [-\tau,0]$, $f : \mathbb{R}_+ \times C_\tau \to \mathbb{H}$ and $g : \mathbb{R}_+ \times C_\tau \to \mathbb{L}_2^0$ are jointly continuous functions, and $\mathbb{W}(t)$ is a $\mathscr{Q}$-Wiener process with values in $\mathbb{K}$. Here, C_τ is the space of continuous functions from $[-\tau,0]$ into $\mathbb{H}$, equipped with the sup norm given by

$$\|z\|_{C_\tau} = \Big(\sup_{-\tau\le\theta\le 0} \|z(\theta)\|^2\Big)^{1/2}.$$

Such an equation is called a stochastic autonomous differential equation with finite delay.

In what follows, in addition to $(3H)_0$ we require the following assumptions:

$(3H)_3$ The coefficients $f(\cdot,\,\cdot)$ and $g(\cdot,\,\cdot)$ satisfy the following Lipschitz and linear growth conditions: there exist positive constants K_i $(i=1,2,3)$ such that the following conditions are satisfied:

$$\begin{aligned}
&\|f(t,x) - f(t,y)\| \le K_1\, \|x-y\|_{C_\tau},\\
&\|g(t,x) - g(t,y)\|_{\mathbb{L}_2^0} \le K_2\, \|x-y\|_{C_\tau},\\
&\|f(t,x)\| + \|g(t,x)\|_{\mathbb{L}_2^0} \le K_3\Big(1 + \|x\|_{C_\tau}\Big),
\end{aligned}$$

for all $x,\, y \in C_\tau$.

Definition 3.35. A stochastic process $\{X(t),\, t \ge 0\}$ is said to be a mild solution of (3.42) if

(i) $X(t)$ is adapted to $\mathscr{F}_t$;

(ii) $X(t)$ is continuous in t almost sure;
(iii) X is measurable with $\int_0^T \|X(t)\|^2\, dt < \infty$ almost surely for any $T > 0$, and

$$X(t;\varphi) = T(t)\varphi(0) + \int_0^t T(t-s)f(s,X_s)\,ds + \int_0^t T(t-s)g(s,X_s)\,d\mathbb{W}(s)$$

for all $t \geq 0$ with probability one;
(iv) $X(t) = \varphi(t)$, $-\tau \leq t \leq 0$, almost surely.

Here, since we are now mainly interested in the exponential stability of the mild solution to (3.42), we introduce the notion of such stability. For the purposes of the stability of (3.42), we shall assume that

$$f(t,0) = g(t,0) \equiv 0 \ \text{ for any } \ t \geq 0\,,$$

so that (3.42) admits a trivial solution when $\varphi \equiv 0$.

We denote a global mild solution of (3.42) corresponding to $\varphi \in C_\tau$ by $X(t;\varphi)$, if one should exist.

Definition 3.36. $X(t;\varphi)$ is said to be *exponentially stable in mean square* if there is a pair of positive constants λ and C such that, for any initial value $\varphi \in C_\tau$,

$$\mathbf{E}\big\|X_t(\cdot;\varphi)\big\|_{C_\tau}^2 \leq C\,\|\varphi\|_{C_\tau}^2 e^{-\lambda t} \ \text{ for all t} \geq 0\,.$$

We obtain the following well-known theorem.

Theorem 3.7. *Suppose that the assumptions $(3H)_0$–$(3H)_3$ hold. Then the mild solution $X(t;\varphi)$ of (3.42) is exponentially stable in mean square whenever the positive constants K_i $(i = 1,2,3)$ are small enough.*

Proof. The proof of the existence and uniqueness of a mild solution of (3.42) is omitted. It can be obtained by the well-known Picard iteration. For the sake of clarity and completeness, the proof of the exponential stability of the solution is reproduced here even though many authors (e.g., Keck and McKibben [110], Luo [133]) obtained the stability with a more general equation than (3.42).

For a fixed $t \geq 0$, we have

$$\begin{aligned}
&\mathbf{E}\big\|X_t(\cdot;\varphi)\big\|_{C_\tau}^2 \\
&\leq 3 \sup_{-\tau\leq\theta\leq 0}\Bigg[\|T(t+\theta)\varphi(0)\|^2 + 3\,\mathbf{E}\bigg\|\int_0^{t+\theta} T(t+\theta-s)f(s,X_s)\,ds\bigg\|^2 \\
&\quad +3\,\mathbf{E}\bigg\|\int_0^{t+\theta} T(t+\theta-s)g(s,X_s)\,d\mathbb{W}(s)\bigg\|^2\Bigg] \\
&= \mathrm{I}_1' + \mathrm{I}_2' + \mathrm{I}_3'\,.
\end{aligned}$$

Using $(3H)_0$ yields

$$\mathrm{I}_1' \le 3\mathrm{M}^2 \big\|\varphi(0)\big\|^2 \sup_{-\tau\le\theta\le 0} \big[\mathrm{e}^{-\delta(\mathrm{t}+\theta)}\big] \le 3\mathrm{M}^2 \big\|\varphi(0)\big\|^2 \mathrm{e}^{-\delta(\mathrm{t}-\tau)}\,.$$

Next, using again $(3H)_0$, an application of the Cauchy–Schwarz inequality, followed by $(3H)_3$, gives us

$$\begin{aligned}
\mathrm{I}_2' &\le 3\,M^2 \sup_{-\tau\le\theta\le 0} \mathbf{E}\Big[\Big(\int_0^{t+\theta} e^{-\delta(t+\theta-s)} \big\|f(s,X_s)\big\|\Big)^2\Big] \\
&\le 3\,M^2 \sup_{-\tau\le\theta\le 0} \Big[\Big\{\Big(\int_0^{t+\theta} e^{-\delta(t+\theta-s)}\,ds\Big)\Big(\int_0^{t+\theta} e^{-\delta(t+\theta-s)}\mathbf{E}\big\|f(s,X_s)\big\|^2\,ds\Big)\Big\}\Big] \\
&\le 3\,\frac{M^2}{\delta} \sup_{-\tau\le\theta\le 0} \int_0^{t+\theta} e^{-\delta(t+\theta-s)}\mathbf{E}\big\|f(s,X_s)\big\|^2\,ds \\
&\le 3\,\frac{M^2}{\delta} K_1^2 \int_0^t e^{-\delta(t-s)}\mathbf{E}\big\|X_s\big\|_{C_\tau}^2\,.
\end{aligned}$$

As to I_3, in a similar manner (with the additional help of Itô isometry), we have

$$\begin{aligned}
\mathrm{I}_3' &= 3\cdot \sup_{-\tau\le\theta\le 0} \Big[\int_0^{t+\theta} \big\|T(t+\theta-s)\big\|^2 \mathbf{E}\big\|g(s,X_s)\big\|_{\mathbb{L}_2^0}^2\,ds\Big] \\
&\le 3\cdot M^2 \sup_{-\tau\le\theta\le 0} \Big[\int_0^{t+\theta} e^{-2\delta(t+\theta-s)}\mathbf{E}\big\|g(s,X_s)\big\|_{\mathbb{L}_2^0}^2\,ds\Big] \\
&\le 3\cdot K_2^2\cdot M^2 \int_0^t e^{-\delta(t-s)}\,\mathbf{E}\big\|X_s\big\|_{C_\tau}^2\,ds\,.
\end{aligned}$$

Combining, we conclude that

$$\begin{aligned}
\mathbf{E}\big\|X_t\big\|_{C_\tau}^2 &\le 3M^2\big\|\varphi(0)\big\|^2 e^{-\delta(t-\tau)} \\
&\quad +3M^2\Big(\frac{K_1^2}{\delta}+K_2^2\Big)\int_0^t e^{-\delta(t-s)}\mathbf{E}\big\|X_s\big\|_{C_\tau}^2\,ds\,.
\end{aligned}\tag{3.43}$$

Now, taking arbitrarily ξ with $0<\xi<\delta$ and $T>0$ large enough, we obtain that

$$\begin{aligned}
&\int_0^T e^{\xi t}\mathbf{E}\big\|X_t\big\|_{C_\tau}^2\,dt \\
&\qquad \le 3M^2\big\|\varphi(0)\big\|^2 e^{\delta\tau}\int_0^T e^{-(\delta-\xi)t}dt \\
&\qquad +3M^2\Big(\frac{K_1^2}{\delta}+K_2^2\Big)\int_0^T e^{-(\delta-\xi)t}\int_0^t e^{\delta s}\mathbf{E}\big\|X_s\big\|_{C_\tau}^2\,ds\,dt\,.
\end{aligned}\tag{3.44}$$

On the other hand,

$$\begin{aligned}\int_0^T e^{-(\delta-\xi)t}\int_0^t e^{\delta s}\mathbf{E}\|X_s\|_{C_\tau}^2\,ds\,dt &= \int_0^T\int_s^T e^{\delta s}\mathbf{E}\|X_s\|_{C_\tau}^2 e^{-(\delta-\xi)t}\,dt\,ds\\ &= \int_0^T e^{\delta s}\mathbf{E}\|X_s\|_{C_\tau}^2\Big\{\int_s^T e^{-(\delta-\xi)t}\,dt\Big\}\,ds\\ &\le \frac{1}{\delta-\xi}\int_0^T e^{\xi s}\mathbf{E}\|X_s\|_{C_\tau}^2\,ds\,. \end{aligned}\tag{3.45}$$

Substituting (3.45) into (3.44) gives

$$\begin{aligned}\int_0^T e^{\xi t}\mathbf{E}\|X_t\|_{C_\tau}^2\,dt \le 3M^2\|\varphi(0)\|^2 e^{\delta\tau}\int_0^T e^{-(\delta-\xi)t}dt\\ +\frac{K_4}{\delta-\xi}\int_0^T e^{\xi s}\mathbf{E}\|X_s\|_{C_\tau}^2\,ds\,,\end{aligned}\tag{3.46}$$

where

$$K_4 = 3M^2\Big(\frac{K_1^2}{\delta}+K_2^2\Big)\,.$$

Since K_4 can be small enough by assumption, it is possible to choose a suitable ξ with $0<\xi<\delta-\frac{K_4}{\delta-\xi}$ such that

$$1-\frac{K_4}{\delta-\xi}>0\,.$$

Hence, letting $T\to\infty$ in (3.46) yields

$$\begin{aligned}\int_0^\infty e^{\xi t}\mathbf{E}\|X_t\|_{C_\tau}^2\,dt &\le \frac{1}{1-\frac{K_4}{\delta-\xi}}\Big[3M^2\|\varphi(0)\|^2 e^{\delta\tau}\int_0^\infty e^{-(\delta-\xi)t}dt\Big]\\ &\le K(\delta,\xi)\,\|\varphi\|_{C_\tau}^2\,.\end{aligned}$$

Therefore, we can deduce from (3.43) that

$$\begin{aligned}\mathbf{E}\|X_t\|_{C_\tau}^2 &\le 3M^2\|\varphi(0)\|^2 e^{-\xi(t-\tau)}+K_4\,e^{-\xi t}\int_0^t e^{\xi s}\mathbf{E}\|X_s\|_{C_\tau}^2\,ds\\ &\le \Big[3M^2\|\varphi(0)\|^2 e^{\xi\tau}+K_4\cdot K(\delta,\xi)\|\varphi\|_{C_\tau}^2\Big]\,e^{-\xi t}\\ &\le K'(\delta,\xi)\,\|\varphi\|_{C_\tau}^2 e^{-\xi t}\,,\end{aligned}$$

as desired.

3.6 Bibliographical Notes

In this chapter, we began by recalling some elementary definitions on probability theory which can be found in any good textbook on probability. The material presented here on sequence of events, random variables, convergence of random variables, and conditional expectation was mostly taken from Billingsley [29], Casella and Berger [34], Grigoriu [82], Mikosch [142], and Pfeiffer [154]. This enabled us to introduce the theory of stochastic processes. The latter is based on non elementary facts from measure theory and classical functional analysis. The concept of martingales was also discussed. The martingales constitute an important class of stochastic processes. The subsections on continuity, separability, measurability, stopping times, Gaussian processes, and martingales were taken from Bakstein and Capasso [15], Bauer [17], Grigoriu [82], Kallianpur [107], Métivier [140], and Mikosch [142]. The Itô integral was subsequently introduced. Its definition goes back to Itô (1942–1944) who introduced the stochastic integral with a random integrand. In 1953, Doob made the connection of Itô integration and martingale theory. Itô integration plays a key role in constructing solutions of stochastic differential equations. The presentation on Itô integration given here follows closely that of Øksendal [150], Mao and Yuan [135], and Da Prato and Zabczyk [47]. In addition, stochastic convolution integrals were introduced. They play an essential role in the construction of stochastic partial differential equations involving sectorial operators. The material used in our presentation on stochastic convolution integrals was taken from Seidler [163]and Seidler and Sobukawa [164]. The concept of Itô integral led us to study stochastic differential equations in a separable Hilbert space. Stochastic calculus discussed in this chapter remains valid in this space. The investigation for stochastic differential equations has attracted considerable attention of researchers. Recently, many authors have studied existence and uniqueness, boundedness, stability, and other quantitative and qualitative properties of solutions to stochastic differential equations. One of the techniques to discuss these topics is the semigroup approach. Many important results have been reported; see for instance Appleby [9], Caraballo and Kai Liu [33], Fu [77], Ichikawa [104], Keck and McKibben [110], Luo [133, 134], Kai Liu [106]. The material used in our presentation on stochastic differential equations was collected from those sources.

Chapter 4
P-th Mean Almost Periodic Random Functions

The concept of almost periodicity is important in probability especially for investigations on stochastic processes [11, 46, 66, 109, 167, 172]. The interest in such a notion lies in its significance and applications arising in engineering, statistics, etc.

The concept of almost periodicity for stochastic processes, which is one of the central questions to be treated in this book, was first introduced in the literature in late 1930s by Slutsky [166], who then obtained reasonable sufficient conditions for sample paths of a stationary process to be almost periodic in the sense of Besicovitch, that is, B^2-almost periodic. A few decades later, two other investigations on the almost periodicity of sample paths followed the pioneer work of Slutsky. Indeed, Udagawa [173] investigated sufficient conditions for sample paths to be almost periodic in the sense of Stepanov, and Kawata [109] studied the uniform almost periodicity of samples paths. A decade ago, Swift [167] extended Kawata results within the framework of harmonizable stochastic processes. Namely, Swift made extensive use of the concept of uniform almost periodicity similar to the one studied by Kawata, to obtain some sufficient conditions for harmonizable stochastic processes to be almost periodic.

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space. In this chapter, we introduce and develop the notion of *p-th mean* almost periodicity. Among others, it will be shown that each p-th mean almost periodic process is uniformly continuous and stochastically bounded [132]. Furthermore, the collection of all p-th mean almost periodic processes is a Banach space when it is equipped with its natural norm. Moreover, we also establish two composition results for p-th mean almost periodic processes (Theorems 4.4 and 4.5). In the next chapters, basic results on p-th mean almost periodic processes, especially Theorems 4.4 and 4.5, will be, subsequently, utilized to study the existence (and uniqueness) of p-th mean almost periodic solutions to various stochastic differential equations on $L^p(\Omega, \mathbb{H})$ where $\mathbb{H}$ is a real separable Hilbert space.

One should point out that several contributions on the study of almost periodic solutions to stochastic differential equations can be found in the literature, see, e.g., [11, 46, 172].

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4.1 Almost Periodic Functions

4.1.1 Introduction

First of all, let us mention that most of the material on almost periodic functions presented here is taken from the book by Diagana [51]. Obviously, there is a vast literature on almost periodic functions. Here we chose, for convenience, to use the concept of almost periodicity in the sense of H. Bohr (1887–1951), which is equivalent to the other classical definitions. For more on almost periodic functions, we refer to the landmark books by Bohr [32], Corduneanu [42], and Fink [73].

4.1.2 Basic Definitions

If $(\mathscr{B}, \|\cdot\|)$ is a Banach space, then $C(\mathbb{R},\mathscr{B})$ will stand for the collection of continuous functions from $\mathbb{R}$ in $\mathscr{B}$. As usual, $BC(\mathbb{R},\mathscr{B})$, the space of all bounded continuous functions from $\mathbb{R}$ into $\mathscr{B}$ introduced in Chapter 1, will be equipped with the sup norm. Similarly, $BC(\mathbb{R}\times\mathscr{B})$ denotes the space of all bounded continuous functions from $\mathbb{R}\times\mathscr{B}$ in $\mathscr{B}$.

Definition 4.1. A function $f \in C(\mathbb{R},\mathscr{B})$ is called (Bohr) almost periodic if for each $\varepsilon > 0$, there exists $T_0(\varepsilon) > 0$ such that every interval of length $T_0(\varepsilon)$ contains a number τ with the following property:

$$\big\|f(t+\tau) - f(t)\big\| < \varepsilon \quad \text{for each } t \in \mathbb{R}.$$

The number τ above is then called an *ε-translation number* of f, and the collection of such functions will be denoted $AP(\mathscr{B})$.

It is well-known that if $f \in AP(\mathscr{B})$, then its mean defined by

$$\mathscr{M}(f) := \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} f(t)dt$$

exists [32]. Consequently, for every $\lambda \in \mathbb{R}$, the following limit

$$a(f,\lambda) := \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} f(t)e^{-i\lambda t}dt$$

exists and is called the Bohr transform of f.

It is well-known that $a(f,\lambda)$ is nonzero at most at countably many points [32].

The set defined by

$$\sigma_b(f) := \Big\{\lambda \in \mathbb{R} : a(f,\lambda) \neq 0\Big\}$$

is called the Bohr spectrum of f [128].

Furthermore, the following approximation theorem is well-known:

Theorem 4.1. (Approximation Theorem) *[132, 128] Let $f \in AP(\mathscr{B})$. Then for every $\varepsilon > 0$ there exists a trigonometric polynomial*

$$P_\varepsilon(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$$

where $a_k \in \mathscr{B}$ and $\lambda_k \in \sigma_b(f)$ such that $\|f(t) - P_\varepsilon(t)\| < \varepsilon$ for all $t \in \mathbb{R}$.

We also have the following properties of the mean:

Proposition 4.1. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be almost periodic functions and let $\alpha \in \mathbb{C}$. Then*
(i) $\mathscr{M}(\overline{f}(t)) = \overline{M(f(t))}$;
(ii) $\mathscr{M}(\alpha f(t)) = \alpha M(f(t))$;
(iii) $\mathscr{M}(f(t)) \geq 0$ whenever $f \geq 0$;
(iv) $\mathscr{M}(f(t) + g(t)) = \mathscr{M}(f(t)) + \mathscr{M}(g(t))$.

Furthermore, if $(f_n(t))$ is a uniformly convergent sequence of almost periodic functions which converges to $f(t)$, then

$$\lim_{n\to\infty} \mathscr{M}(f_n(t)) = \mathscr{M}(f(t)).$$

Proof. The proof is left as an exercise.

Example 4.1. (i) Each periodic function $\varphi : \mathbb{R} \to \mathscr{B}$ is almost periodic.

(ii) The function $f_\alpha(t) = \sin t + \sin t\alpha$ where $\alpha \in \mathbb{R} - \mathbb{Q}$, is a classical example of an almost periodic function on $\mathbb{R}$, which is not periodic.

(iii) Any trigonometric polynomial in the form $P(t) = \sum_{k=0}^{N} a_k e^{is_k t}$ where $a_k \in \mathscr{B}$ and $s_k \in \mathbb{R}$ for $k = 0, 1, \ldots, N$, belongs to $AP(\mathscr{B})$.

Remark 4.1. Let $f, g : \mathbb{R} \to \mathscr{B}$ be almost periodic functions and let $\alpha \in \mathbb{R}$. Then the following hold:
(i) $f + g$ is almost periodic; if f, g are $\mathbb{C}$-valued, then $f.g$ is also almost periodic.
(ii) $t \mapsto f(t+\alpha)$, $t \mapsto f(\alpha t)$, and $t \mapsto \alpha f(t)$ are almost periodic.
(iii) Each almost periodic function is bounded.

4.1.3 Properties of Almost Periodic Functions

Proposition 4.2. *[61] If $f : \mathbb{R} \to \mathscr{B}$ is almost periodic, then f is uniformly continuous in $t \in \mathbb{R}$. Moreover, the range $R(f) = \Big\{f(t) : t \in \mathbb{R}\Big\}$ is precompact in $\mathscr{B}$.*

Proof. First of all, notice that every trigonometric polynomial is uniformly continuous. Let $f \in AP(\mathscr{B})$. According to Theorem 6.16, for every $\varepsilon > 0$ there exists a $\mathscr{B}$-valued trigonometric polynomial P_ε such that

$$\|f(t) - P_\varepsilon(t)\| < \frac{\varepsilon}{3}$$

for all $t \in \mathbb{R}$.

Now from the uniform continuity of P_ε, there exists $\delta_\varepsilon > 0$ such that

$$\|P_\varepsilon(t_1) - P_\varepsilon(t_2)\| < \frac{\varepsilon}{3}, \ \ |t_1 - t_2| < \delta_\varepsilon.$$

From

$$\|f(t_1) - f(t_2)\| \le \|f(t_1) - P_\varepsilon(t_1)\| + \|P_\varepsilon(t_1) - P_\varepsilon(t_2)\| + \|P_\varepsilon(t_2) - f(t_2)\|$$

it follows that

$$\|f(t_1) - f(t_2)\| < \varepsilon, \ \ |t_1 - t_2| < \delta_\varepsilon,$$

and hence f is uniformly continuous.

It remains to show that the range $R(f) = \Big\{ f(t) : t \in \mathbb{R} \Big\}$ is precompact in $\mathscr{B}$. Here again, from the almost periodicity of f, for each $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the following property:

$$\|f(t+\tau) - f(t)\| < \frac{\varepsilon}{2}, \ \ \forall t \in \mathbb{R}.$$

Since $f([0, l_\varepsilon])$ is compact in $\mathscr{B}$, let us choose a finite sequence $t_1, ..., t_n \in [0, l_\varepsilon]$ such that

$$f(t) \in \bigcup_{i=1}^{n} B(f(t_i), \frac{\varepsilon}{2}),$$

where $B(x, r) = \{y \in \mathscr{B} : \|y - x\| \le r\}$.

Now let $t \in \mathbb{R}$, $\tau = \tau(t)$ such that $0 < t + \tau < l_\varepsilon$, t_j an element of the sequence $t_1, ..., t_n$ such that $f(t+\tau) \in B(f(t_i), \frac{\varepsilon}{2})$. Then

$$\|f(t) - f(t_j)\| \le \|f(t) - f(t+\tau)\| + \|f(t+\tau) - f(t_j)\| < \varepsilon$$

so that

$$R(f) \subseteq \bigcup_{i=1}^{n} B(f(t_i), \frac{\varepsilon}{2}).$$

Thus from the arbitrariness of ε it follows that $R(f)$ is precompact.

Proposition 4.3. *[61] Let $f \in AP(\mathbb{R})$. If $g \in L^1(\mathbb{R})$, then $f * g$, the convolution of f with g on $\mathbb{R}$, is almost periodic.*

Proof. Since f is continuous and $g \in L^1(\mathbb{R})$, it is not hard to see that the function $t \mapsto (f * g)(t)$ is continuous. Moreover,

$$|(f * g)(t)| \le \|f\|_\infty \|g\|_1$$

for each $t \in \mathbb{R}$, where $\|g\|_1$ is the L^1-norm of g, and therefore $f * g \in BC(\mathbb{R}, \mathbb{R})$.

It remains to prove that $f * g$ is almost periodic. First of all, note that when $g \equiv 0$ there is nothing to prove. From now on, we suppose $g \not\equiv 0$.

Since $f \in AP(\mathbb{R})$, for every $\varepsilon > 0$ there exists $T_0(\varepsilon) > 0$ such that for all $\delta \in \mathbb{R}$ there exists $\tau \in [\delta, \delta + T_0(\varepsilon)]$ with

$$|f(\sigma+\tau) - f(\sigma)| < \frac{\varepsilon}{\|g\|_1} \ \text{ for each } \ \sigma \in \mathbb{R}.$$

In particular, the following holds:

$$|f(t-s+\tau) - f(t-s)| < \frac{\varepsilon}{\|g\|_1} \ \text{ for each } \sigma = t-s \in \mathbb{R}. \tag{4.1}$$

Now

$$(f*g)(t+\tau) - (f*g)(t) = \int_{-\infty}^{+\infty} \{f(t-\sigma+\tau) - f(t-\sigma)\} g(\sigma) d\sigma, \ \ t \in \mathbb{R}.$$

Thus from Eq. (4.1) and the fact that $g \in L^1(\mathbb{R})$ it easily follows that

$$\|\sigma_\tau(f*g) - (f*g)\|_\infty < \varepsilon,$$

and hence $f * g \in AP(\mathbb{R})$.

Proposition 4.4. *[61] If $f, g : \mathbb{R} \to \mathbb{C}$ are almost periodic functions and if there exists a constant $m > 0$ such that*

$$m \leq \big|g(t)\big|$$

for each $t \in \mathbb{R}$, then $(f/g)(t) = f(t)/g(t)$ is almost periodic.

Proof. It is sufficient to prove that $1/g(t)$ is almost periodic. Indeed, for each $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the following property:

$$|g(t+\tau) - g(t)| < m^2 \varepsilon \quad (t \in \mathbb{R}).$$

Now

$$\begin{aligned} \left| \frac{1}{g(t+\tau)} - \frac{1}{g(t)} \right| &= \frac{|g(t+\tau) - g(t)|}{|g(t+\tau) g(t)|} \\ &\leq \frac{|g(t+\tau) - g(t)|}{m^2} \\ &< \varepsilon. \end{aligned}$$

Proposition 4.5. *[61] Let $(f_n(t))_{n \in \mathbb{N}}$ be a sequence of almost periodic functions such that $f_n(t)$ converges $f(t)$ uniformly in $t \in \mathbb{R}$. Then f is almost periodic.*

Proof. For each $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\|f_n(t) - f(t)\| \leq \frac{\varepsilon}{3}, \ \forall t \in \mathbb{R}, \ n \geq N(\varepsilon).$$

Since $f_N(t)$ is almost periodic, there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the following property:

$$\|f_N(t+\tau) - f_N(t)\| < \frac{\varepsilon}{3}, \ \forall t \in \mathbb{R}.$$

Now

$$\begin{aligned}
\|f(t+\tau) - f(t)\| &= \|f(t+\tau) - f_N(t+\tau) + f_N(t+\tau) - f_N(t) + f_N(t) - f(t)\| \\
&\leq \|f(t+\tau) - f_N(t+\tau)\| + \|f_N(t+\tau) - f_N(t)\| \\
&+ \|f_N(t) - f(t)\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Proposition 4.6. *[61] Let f be an almost periodic function such that f' is uniformly continuous on $\mathbb{R}$, then f' is also almost periodic.*

Proof. Set $f_n(t) = n\left[f(t+\frac{1}{n}) - f(t)\right]$ for each $t \in \mathbb{R}$ for $n = 1, 2, \ldots$. Clearly, $f_n : \mathbb{R} \mapsto \mathbb{C}$ is a sequence of almost periodic functions, which converges uniformly to f' on the line. One then completes the proof by using Proposition 4.5.

Theorem 4.2. *(Bochner's Criterion) A function $f : \mathbb{R} \to \mathscr{B}$ is almost periodic if and only if for every sequence of real numbers $(s_n)_{n \in \mathbb{N}}$ there exists a subsequence $(\sigma_n)_{n \in \mathbb{N}}$ such that $\{f(t+\sigma_n)\}_{n \in \mathbb{N}}$ converges uniformly in $t \in \mathbb{R}$.*

Proof. See the proof in N'Guérékata [147, Proof of Theorem 3.1.8, p. 55].

Definition 4.2. A normed vector space $(\mathscr{B}, \|\cdot\|)$ is said to be uniformly convex if for every $0 < \varepsilon < 2$ there exists a number $\delta(\varepsilon) > 0$ such that if $x, y \in \mathscr{B}$ satisfy

$$\|x\| = \|y\| = 1 \ \text{ and } \ \|x - y\| \geq \varepsilon,$$

then $\|(x+y)/2\| \leq 1 - \delta(\varepsilon)$.

Remark 4.2. (a) $\mathbb{R}^n$ equipped with the Euclidean norm is uniformly convex.
(b) Hilbert spaces are uniformly convex.

Proposition 4.7. *Suppose that the Banach space $\mathscr{B}$ is uniformly convex. If $f : \mathbb{R} \to \mathscr{B}$ is almost periodic, then its antiderivative*

$$F(t) = \int_0^t f(\sigma) d\sigma \tag{4.2}$$

is almost periodic if and only if it is bounded in $\mathscr{B}$, i.e., $\sup_{t \in \mathbb{R}} \|F(t)\| < \infty$.

Proof. See the proof in Corduneanu [42, Proof of Theorem 6.20, p. 179–180].

Definition 4.3. A function $F \in BC(\mathbb{R} \times \mathscr{B})$, $(t,x) \mapsto F(t,x)$ is called almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \Gamma$ ($\Gamma \subset \mathscr{B}$ being a compact subset) if for each $\varepsilon > 0$ there exists $T_0(\varepsilon) > 0$ such that every interval of length $T_0(\varepsilon) > 0$ contains a number τ with the following property:

$$\left\|F(t+\tau,x) - F(t,x)\right\| < \varepsilon, \ \ \forall x \in \Gamma, \forall t \in \mathbb{R}.$$

Here again, the number τ above is called an ε-translation number of F, and the class of such functions will be denoted $AP(\mathbb{R} \times \mathscr{B})$.

Proposition 4.8. *Let $\left(\mathscr{B}, \|\cdot\|\right)$ and $\left(\mathscr{B}', \|\cdot\|'\right)$ be two Banach spaces over the same field $\mathbb{F}$. Let $f : \mathbb{R} \times \mathscr{B} \to \mathscr{B}'$, $(t,x) \to f(t,x)$ be almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathscr{B}$. Suppose that f is Lipschitz in $x \in \mathscr{B}$ uniformly in $t \in \mathbb{R}$, i.e., there exists $L \geq 0$ such that*

$$\left\|f(t,x) - f(t,y)\right\|' \leq L.\left\|x-y\right\|, \ \ \forall x,y \in \mathscr{B}, \ \ t \in \mathbb{R}.$$

If $\phi : \mathbb{R} \to \mathscr{B}$ is almost periodic, then the function $h(t) = f(t,\phi(t)) : \mathbb{R} \to \mathscr{B}'$ is also almost periodic.

Proof. See the proof in Corduneanu [42, Proof of Theorem 2.8, p. 61].

Definition 4.4. A function $f \in BC(\mathbb{R}, \mathscr{B})$ is called (Bochner) almost periodic if for any sequence $(\sigma_n')_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $(\sigma_n)_{n\in\mathbb{N}}$ of $(\sigma_n')_{n\in\mathbb{N}}$ such that the sequence of functions $(f(t+\sigma_n))_{n\in\mathbb{N}}$ converges uniformly in $t \in \mathbb{R}$.

Theorem 4.3. *A function $f \in BC(\mathbb{R}, \mathscr{B})$ is Bohr almost periodic if and only if it is Bochner almost periodic.*

Proof. A detailed proof of this result in $\mathbb{R}$ can be found in Corduneanu [42]. Obviously that proof can be easily extended to a general Banach space $\mathscr{B}$.

4.2 p-th Mean Almost Periodic Processes

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space. As stated in Chapter 2, for $p \geq 2$ the spaces $L^p(\Omega; \mathscr{B})$ and $BUC(\mathbb{R}; L^p(\Omega; \mathscr{B}))$ are Banach spaces when they are equipped with their respective norms $\|\cdot\|_{L^p(\Omega,\mathscr{B})}$ and $\|\cdot\|_\infty$.

Definition 4.5. A stochastic process $X : \mathbb{R} \to L^p(\Omega; \mathscr{B})$ is said to be continuous whenever

$$\lim_{t\to s} \mathbf{E}\left\|X(t) - X(s)\right\|^p = 0.$$

Definition 4.6. A stochastic process $X : \mathbb{R} \to L^p(\Omega;\mathscr{B})$ is said to be stochastically bounded whenever

$$\lim_{N\to\infty} \sup_{t\in\mathbf{R}} \mathbf{P}\Big\{ \big\|X(t)\big\| > N \Big\} = 0.$$

Definition 4.7. A continuous stochastic process $X : \mathbb{R} \to L^p(\Omega;\mathscr{B})$ is said to be p-th mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t\in\mathbf{R}} \mathbf{E}\big\|X(t+\tau) - X(t)\big\|^p < \varepsilon\,. \tag{4.3}$$

A continuous stochastic process X, which is 2-nd mean almost periodic will be called *square-mean almost periodic*.

Like for classical almost periodic functions, the number τ will be called an ε-translation of X.

The collection of all p-th mean almost periodic stochastic processes $X : \mathbb{R} \to L^p(\Omega;\mathscr{B})$ will be denoted by $AP(\mathbb{R};L^p(\Omega;\mathscr{B}))$.

The next lemma provides some properties of p-th mean almost periodic processes.

Lemma 4.1. *If X belongs to $AP(\mathbb{R};L^p(\Omega;\mathscr{B}))$, then*
(i) the mapping $t \to \mathbf{E}\big\|X(t)\big\|^p$ is uniformly continuous;
(ii) there exists a constant $M > 0$ such that $\mathbf{E}\big\|X(t)\big\|^p \le M$, for each $t \in \mathbb{R}$;
(iii) X is stochastically bounded.

Proof. The proofs for (i) and (ii) are not difficult and hence are left as an exercise.

To prove (iii), we combine both Chebychev's inequality and (ii) to obtain

$$\sup_{t\in\mathbb{R}} \mathbf{P}\Big\{ \|X(t)\| > N \Big\} \le \frac{1}{N^p} \sup_{t\in\mathbb{R}} \mathbf{E}\big\|X(t)\big\|^p \le \frac{M}{N^p},$$

and hence

$$\lim_{N\to\infty} \sup_{t\in\mathbb{R}} \mathbf{P}\Big\{ \big\|X(t)\big\| > N \Big\} = 0.$$

Lemma 4.2. *$AP(\mathbb{R};L^p(\Omega;\mathscr{B})) \subset BUC(\mathbb{R};L^p(\Omega;\mathscr{B}))$ is a closed subspace.*

In view of Lemma 4.2, it follows that the space $AP(\mathbb{R};L^p(\Omega;\mathscr{B}))$ of p-th mean almost periodic processes equipped with the sup norm $\big\|\cdot\big\|_\infty$ is a Banach space.

Let $\big(\mathscr{B}_1, \big\|\cdot\big\|_1\big)$ and $\big(\mathscr{B}_2, \big\|\cdot\big\|_2\big)$ be Banach spaces and let $L^p(\Omega;\mathscr{B}_1)$ and $L^p(\Omega;\mathscr{B}_2)$ be their corresponding L^p-spaces, respectively.

Definition 4.8. A function $F : \mathbb{R}\times L^p(\Omega;\mathscr{B}_1) \to L^p(\Omega;\mathscr{B}_2)$, $(t,Y) \mapsto F(t,Y)$, which is jointly continuous, is said to be p-th mean almost periodic in $t \in \mathbb{R}$ uniformly in $Y \in K$ where $K \subset L^p(\Omega;\mathscr{B}_1)$ is compact if for any $\varepsilon > 0$, there exists $l_\varepsilon(K) > 0$ such that any interval of length $l_\varepsilon(K)$ contains at least a number τ for which

$$\sup_{t\in\mathbf{R}} \mathbf{E}\big\|F(t+\tau,Y) - F(t,Y)\big\|_2^p < \varepsilon$$

for each stochastic process $Y : \mathbb{R} \to K$.

4.2.1 Composition of p-th Mean Almost Periodic Processes

We have the following composition results. We prove Theorem 4.5 only and leave the proof of Theorem 4.4 as an exercise.

Theorem 4.4. *Let $F : \mathbb{R} \times L^p(\Omega;\mathscr{B}_1) \to L^p(\Omega;\mathscr{B}_2)$, $(t,Y) \mapsto F(t,Y)$ be a p-th mean almost periodic process in $t \in \mathbb{R}$ uniformly in $Y \in K$, where $K \subset L^p(\Omega;\mathscr{B}_1)$ is compact. Suppose that F is Lipschitzian in the following sense:*

$$\mathbf{E}\big\|F(t,Y) - F(t,Z)\big\|_2^p \le M\,\mathbf{E}\big\|Y - Z\big\|_1^p$$

for all $Y,Z \in L^p(\Omega;\mathscr{B}_1)$ and for each $t \in \mathbb{R}$, where $M > 0$. Then for any p-th mean almost periodic process $\Phi : \mathbb{R} \to L^p(\Omega;\mathscr{B}_1)$, the stochastic process $t \mapsto F(t,\Phi(t))$ is p-th mean almost periodic.

Proof. The proof is similar to that of Proposition 4.8 and hence omitted.

Theorem 4.5. *Let $F : \mathbb{R} \times L^p(\Omega;\mathscr{B}_1) \to L^p(\Omega;\mathscr{B}_2)$, $(t,Y) \mapsto F(t,Y)$ be a p-th mean almost periodic process in $t \in \mathbb{R}$ uniformly in $Y \in K$, where $K \subset L^p(\Omega;\mathscr{B}_1)$ is any compact subset. Suppose that $F(t,\cdot)$ is uniformly continuous on bounded subsets $K' \subset L^p(\Omega;\mathscr{B}_1)$ in the following sense: for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $X,Y \in K'$ and $\mathbf{E}\big\|X - Y\big\|_1^p < \delta_\varepsilon$, then*

$$\mathbf{E}\big\|F(t,Y) - F(t,Z)\big\|_2^p < \varepsilon, \ \ \forall t \in \mathbb{R}.$$

Then for any p-th mean almost periodic process $\Phi : \mathbb{R} \to L^p(\Omega;\mathscr{B}_1)$, the stochastic process $t \mapsto F(t,\Phi(t))$ is p-th mean almost periodic.

Proof. Since $\Phi : \mathbb{R} \to L^p(\Omega;\mathscr{B}_1)$ is a p-th mean almost periodic process, for all $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that every interval of length $l_\varepsilon > 0$ contains a τ with the property that

$$\mathbf{E}\big\|\Phi(t+\tau) - \Phi(t)\big\|_1^p < \varepsilon, \ \ \forall t \in \mathbb{R}. \tag{4.4}$$

In addition, $\Phi : \mathbb{R} \to L^p(\Omega;\mathscr{B}_1)$ is bounded, that is, $\sup\limits_{t\in R}\mathbf{E}\big\|\Phi(t)\big\|_1^p < \infty$. Let $K'' \subset L^p(\Omega;\mathscr{B}_1)$ be a bounded subset such that $\Phi(t) \in K''$ for all $t \in \mathbb{R}$.

Now

$$\begin{aligned}\mathbf{E}\big\|F(t+\tau,\Phi(t+\tau)) - F(t,\Phi(t))\big\|_2^p &\le 2^{p-1}\mathbf{E}\big\|F(t+\tau,\Phi(t+\tau)) - F(t+\tau,\Phi(t))\big\|_2^p \\ &\quad + 2^{p-1}\mathbf{E}\big\|F(t+\tau,\Phi(t)) - F(t,\Phi(t))\big\|_2^p.\end{aligned}$$

Taking into account Eq. (4.4) (take $\delta_\varepsilon = \varepsilon$) and using the uniform continuity of F on bounded subsets of $L^p(\Omega;\mathscr{B}_1)$ it follows that

$$\sup_{t\in\mathbb{R}} \mathbf{E}\big\|F(t+\tau,\Phi(t+\tau)) - F(t+\tau,\Phi(t))\big\|_2^p < \frac{\varepsilon}{2^p}. \tag{4.5}$$

Similarly, using the p-th mean almost periodicity of F it follows that

$$\sup_{t\in\mathbb{R}} \mathbf{E}\big\|F(t+\tau,\Phi(t)) - F(t,\Phi(t))\big\|_2^p < \frac{\varepsilon}{2^p}. \tag{4.6}$$

Combining (4.5) and (4.6) one obtains that

$$\sup_{t\in\mathbb{R}} \mathbf{E}\big\|F(t+\tau,\Phi(t+\tau)) - F(t,\Phi(t))\big\|_2^p < \varepsilon,$$

and hence the stochastic process $t \mapsto F(t,\Phi(t))$ is p-th mean almost periodic.

4.3 Bibliographical Notes

The classical results on almost periodic functions, p-th mean almost periodic processes, and some of their proofs found in this chapter are mainly taken from Bohr [32], Corduneanu [42], Diagana [61], or the recent papers of Bezandry and Diagana [20, 21, 22, 23, 25, 26, 27].

In a recent paper by Fu and Liu [76], the concept of square-mean almost automorphy was introduced. Such a notion generalizes in a natural fashion the notion of square-mean almost periodicity, which has been studied in various situations by Bezandry and Diagana [20, 21, 22, 23, 24]. In [76], the authors made use of the Banach fixed principle to obtain the existence of a square-mean almost automorphic solution to the autonomous stochastic differential equations. Similarly, Liang *et al.* [120, 121, 181, 182] introduced the concept of pseudo almost automorphy, which is a powerful generalization of both the notion of almost automorphy due to Bochner [31] and that of pseudo almost periodicity due to Zhang (see [61] for instance). Such a concept has recently generated several developments. Motivated by these papers, Bezandry and Diagana [27] recently introduced some new classes of stochastic processes called respectively p-th mean pseudo almost automorphic stochastic processes and p-th mean pseudo almost periodic stochastic processes for $p \geq 2$.

It should be mentioned that the notion of p-th mean pseudo almost automorphy generalizes in a natural fashion both the notion of square-mean almost periodicity and that of square-mean almost automorphy. Indeed, a stochastic process $X \in BC(\mathbb{R};L^p(\Omega;\mathscr{B}))$ is called p-th pseudo almost automorphic if it can be expressed as $X = Y + \Phi$, where $Y \in AA(\mathbb{R};L^p(\Omega;\mathscr{B}))$ and $\Phi \in PAP_0(\mathbb{R};L^p(\Omega;\mathscr{B}))$, where $PAP_0(\mathbb{R};L^p(\Omega;\mathscr{B}))$ is the collection of all $X \in BC(\mathbb{R},L^p(\Omega;\mathscr{B}))$ such that

$$\lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\|X(s)\|^p ds\right]^{1/p}=0.$$

Equivalently, $PAP_0(\mathbb{R};L^p(\Omega;\mathscr{B}))$ is the collection of all $X\in BC(\mathbb{R},L^p(\Omega;\mathscr{B}))$ such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{E}\|X(s)\|^p ds=0.$$

Similarly, a stochastic process $X\in BC(\mathbb{R};L^p(\Omega;\mathscr{B}))$ is called p-th pseudo almost periodic if it can be expressed as $X=Y+\Phi$, where $Y\in AP(\mathbb{R};L^p(\Omega;\mathscr{B}))$ and $\Phi\in PAP_0(\mathbb{R};L^p(\Omega;\mathscr{B}))$.

Chapter 5
Existence Results for Some Stochastic Differential Equations

Throughout this chapter, $(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ and $(\mathbb{H}, \|\cdot\|)$ stand for real separable Hilbert spaces and $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$ denotes a filtered probability space. The space $\mathbb{L}_2^0 = L_2(\mathbb{K}, \mathbb{H})$ stands for the space of all $\mathscr{Q}$-Hilbert–Schmidt operators acting from $\mathbb{K}$ into $\mathbb{H}$, equipped with the Hilbert–Schmidt norm $\|\cdot\|_{\mathbb{L}_2^0}$.

5.1 The Autonomous Case

In this section we study the existence and uniqueness of p-th mean almost periodic solutions to the semilinear stochastic differential equation

$$dX(t) = AX(t)\,dt + F(t, X(t))\,dt + G(t, X(t))\,d\mathbb{W}(t),\ t \in \mathbb{R}, \tag{5.1}$$

where $A : D(A) \subset L^p(\Omega; \mathbb{H}) \to L^p(\Omega; \mathbb{H})$ is a closed densely defined (possibly unbounded) linear operator, $F, G : \mathbb{R} \times L^p(\Omega; \mathbb{H}) \mapsto L^p(\Omega; \mathbb{L}_2^0)$ are jointly continuous functions, and $\mathbb{W}$ is a $\mathscr{Q}$-Wiener process with values in $\mathbb{K}$.

In this section, in addition to $(3H)_0$ we require the following additional assumptions:

$(5H)_1$ Let the function $F : \mathbb{R} \times L^p(\Omega; \mathbb{H}) \to L^p(\Omega; \mathbb{H})$ $(t, X) \mapsto F(t, X)$ be p-th mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in \mathscr{O}$ ($\mathscr{O} \subset L^p(\Omega; \mathbb{H})$ being a compact subspace). Moreover, F is Lipschitz in the following sense: there exists $K > 0$ for which

$$\mathbf{E}\big\|F(t, X) - F(t, Y)\big\|^p \leq K\,\mathbf{E}\big\|X - Y\big\|^p$$

for all stochastic processes $X, Y \in L^p(\Omega; \mathbb{H})$ and $t \in \mathbb{R}$.

$(5H)_2$ Let the function $G : \mathbb{R} \times L^p(\Omega; \mathbb{H}) \to L^p(\Omega; \mathbb{L}_2^0)$ $(t, X) \mapsto G(t, X)$ be p-th mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in \mathscr{O}'$ ($\mathscr{O}' \subset L^p(\Omega; \mathbb{H})$ being a compact subspace). Moreover, G is Lipschitz in the following sense: there exists $K' > 0$ for which

$$\mathbf{E}\big\|G(t, X) - G(t, Y)\big\|_{\mathbb{L}_2^0}^p \leq K'\,\mathbf{E}\big\|X - Y\big\|^p$$

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for all stochastic processes $X, Y \in L^p(\Omega; \mathbb{H})$ and $t \in \mathbb{R}$.

Definition 5.1. An $\mathscr{F}_t$-progressively measurable process $\{X(t)\}_{t\in\mathbb{R}}$ is called a mild solution of (5.1) on $\mathbb{R}$ if

$$X(t) = T(t-s)X(s) + \int_s^t T(t-\sigma)F(\sigma, X(\sigma))\, d\sigma + \int_s^t T(t-\sigma)G(\sigma, X(\sigma))\, d\mathbb{W}(\sigma)$$

for all $t \geq s$ for each $s \in \mathbb{R}$.

Using the classical Banach fixed-point principle, we obtain the following:

Theorem 5.1. *Under assumptions* $(3H)_0$-$(5H)_1$-$(5H)_2$*, then Eq. (5.1) has a unique p-th mean almost periodic mild solution, which can be explicitly expressed as follows:*

$$X(t) = \int_{-\infty}^t T(t-\sigma)F(\sigma, X(\sigma))\, d\sigma + \int_{-\infty}^t T(t-\sigma)G(\sigma, X(\sigma))\, d\mathbb{W}(\sigma), \quad t \in \mathbb{R}$$

whenever $\Theta_p < 1$*, where*

$$\Theta_p := 2^p M^p \left[K_F \left(\frac{1}{\delta^p}\right) + C_p K_G \left(\frac{p-2}{p\,\delta}\right)^{\frac{p-2}{2}} \left(\frac{1}{\delta\, p}\right)\right] \textit{ for } \; p > 2$$

and

$$\Theta_p := 2M^2 \left(\frac{K}{\delta^2} + \frac{K'}{\delta}\right) \textit{ for } \; p = 2.$$

Proof. First of all, note that X given by

$$X(t) = \int_{-\infty}^t T(t-\sigma)F(\sigma, X(\sigma))\, d\sigma + \int_{-\infty}^t T(t-\sigma)G(\sigma, X(\sigma))\, d\mathbb{W}(\sigma) \quad (5.2)$$

is well-defined and satisfies

$$\begin{aligned} X(t) &= T(t-s)X(s) + \int_s^t T(t-\sigma)F(\sigma, X(\sigma))\, d\sigma \\ &+ \int_s^t T(t-\sigma)G(\sigma, X(\sigma))\, d\mathbb{W}(\sigma) \end{aligned}$$

for all $t \geq s$ for each $s \in \mathbb{R}$, and hence X given by Eq. (5.2) is a mild solution to Eq. (5.1).

Define $\Lambda X(t) = \Phi X(t) + \Psi X(t)$, where

$$\begin{aligned} \Phi X(t) &= \int_{-\infty}^t T(t-\sigma)F(\sigma, X(\sigma))\, d\sigma, \;\; \Psi X(t) \\ &= \int_{-\infty}^t T(t-\sigma)G(\sigma, X(\sigma))\, d\mathbb{W}(\sigma)\,. \end{aligned}$$

Let us first show that $\Phi X(\cdot)$ and $\Psi X(\cdot)$ are p-th mean almost periodic whenever X is. Indeed, assuming that X is p-th mean almost periodic and using $(5H)_1$, and Theorem 4.4, one can easily see that the mapping $\sigma \mapsto F(\sigma, X(\sigma))$ is p-th mean almost periodic. That is, for each $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that any interval of length l_ε contains at least τ for which

$$\mathbf{E}\big\|F(\sigma+\tau, X(\sigma+\tau)) - F(\sigma, X(\sigma))\big\|^p < \mu\,\varepsilon$$

for each $\sigma \in \mathbb{R}$, with $\mu = \dfrac{\delta^p}{M^p}$.

Using assumption $(3H)_0$ it follows that

$$\begin{aligned}
&\mathbf{E}\big\|\Phi X(t+h) - \Phi X(t)\big\|^p \\
&\le \mathbf{E}\Big[\int_{-\infty}^t \|T(t-\sigma)\|\big\|F(\sigma+\tau, X(\sigma+\tau)) - F(\sigma, X(\sigma))\big\|\,d\sigma\Big]^p \\
&\le M^p\mathbf{E}\Big[\int_{-\infty}^t e^{-\delta(t-\sigma)}\big\|F(\sigma+\tau, X(\sigma+\tau)) - F(\sigma, X(\sigma))\big\|\,d\sigma\Big]^p \\
&\le M^p\mathbf{E}\Big[\int_{-\infty}^t e^{-\frac{1}{q}\delta(t-\sigma)}e^{-\frac{1}{p}\delta(t-\sigma)}\big\|F(\sigma+\tau, X(\sigma+\tau)) - F(\sigma, X(\sigma))\big\|\,d\sigma\Big]^p,
\end{aligned}$$

where $q > 0$ solves $p^{-1} + q^{-1} = 1$.

Now using Hölder's inequality we obtain

$$\begin{aligned}
&\mathbf{E}\big\|\Phi X(t+h) - \Phi X(t)\big\|^p \\
&\le M^p\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}\,d\sigma\right)^{p-1} \times \\
&\quad \times\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}\mathbf{E}\big\|F(\sigma+\tau, X(\sigma+\tau)) - F(\sigma, X(\sigma))\big\|^p\,d\sigma\right) \\
&\le M^p\left(\int_{-\infty}^t e^{-\delta(t-\sigma)}\,d\sigma\right)^{p}\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|F(\sigma+\tau, X(\sigma+\tau)) - F(\sigma, X(\sigma))\big\|^p \\
&\le \frac{M^p}{\delta^p}\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|F(\sigma+\tau, X(\sigma+\tau)) - F(\sigma, X(\sigma))\big\|^p \\
&\le \varepsilon.
\end{aligned}$$

In view of the above, $\mathbf{E}\big\|\Phi X(t+\tau) - \Phi X(t)\big\|^2 < \varepsilon$ for each $t \in \mathbb{R}$, that is, $\Phi X(\cdot)$ is p-th mean almost periodic.

For $\Psi X(\cdot)$, we split the proof in two cases: $p > 2$ and $p = 2$. We first start with the case where $p > 2$. Assuming that X is p-th mean almost periodic and using $(5H)_2$ and Theorem 4.4, one can easily see that $s \mapsto G(\sigma, X(\sigma))$ is p-th mean almost periodic. Therefore, for each $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that any interval of length l_ε contains at least τ for which

$$\mathbf{E}\big\|G(\sigma+\tau, X(\sigma+\tau)) - G(\sigma, X(\sigma))\big\|^p_{\mathbb{L}_2^0} < \frac{\varepsilon}{C^p\, M^p\left(\frac{2}{\delta\, p}\right)\left(\frac{p-2}{p\,\delta}\right)^{\frac{p-2}{2}}}$$

for each $\sigma \in \mathbb{R}$.

The next step consists in proving the p-th mean almost periodicity of $\Psi X(\cdot)$. Of course, this is a bit more complicated than the previous case because of the involvement of the Brownian motion $\mathbb{W}$. To overcome such a difficulty, we make extensive use of Proposition 3.23 and the properties of $\tilde{\mathbb{W}}$ defined by $\tilde{\mathbb{W}}(s) := \mathbb{W}(s+\tau) - \mathbb{W}(\tau)$ for each s. Note that $\tilde{\mathbb{W}}$ is also a Brownian motion and has the same distribution as $\mathbb{W}$.

Using assumption $(3H)_0$, Hölder's inequality, and Proposition 3.23, we have

$$\begin{aligned}
&\mathbf{E}\big\|\Psi X(t+\tau) - \Psi X(t)\big\|^p \\
&\leq C_p \mathbf{E}\Big[\int_{-\infty}^t \big\|T(t-\sigma)\big\|^2 \big\|G(\sigma+\tau, X(\sigma+\tau)) - G(\sigma, X(\sigma))\big\|_{\mathbb{L}_2^0}^2 \, d\sigma\Big]^{p/2} \\
&\leq C_p M^p \mathbf{E}\Big[\int_{-\infty}^t e^{-2\delta(t-s)} \big\|G(\sigma+\tau, X(\sigma+\tau)) - G(\sigma, X(\sigma))\big\|_{\mathbb{L}_2^0}^2 \, d\sigma\Big]^{p/2} \\
&\leq C_p M^p \Big(\int_{-\infty}^t e^{\frac{p}{p-2}\delta(t-s)} \, d\sigma\Big)^{\frac{p-2}{2}} \times \\
&\quad \times \Big(\int_{\infty}^t e^{-\frac{p}{2}\delta(t-s)} \mathbf{E}\big\|G(\sigma+\tau, X(\sigma+\tau)) - G(\sigma, X(\sigma))\big\|_{\mathbb{L}_2^0}^p \, d\sigma\Big) \\
&\leq C_p M^p \Big(\int_{-\infty}^t e^{-\frac{p\delta}{p-2}(t-\sigma)} \, d\sigma\Big)^{\frac{p-2}{2}} \Big(\int_{\infty}^t e^{-\frac{p\delta}{2}\delta(t-\sigma)} \, d\sigma \times \\
&\quad \times \sup_{\sigma\in\mathbb{R}} \mathbf{E}\big\|G(\sigma+\tau, X(\sigma+\tau)) - G(\sigma, X(\sigma))\big\|_{\mathbb{L}_2^0}^p \, d\sigma\Big) \\
&\leq C^p M^p \Big(\frac{2}{\delta p}\Big) \Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}} \sup_{\sigma\in\mathbb{R}} \mathbf{E}\big\|G(\sigma+\tau, X(\sigma+\tau)) - G(\sigma, X(\sigma))\big\|_{\mathbb{L}_2^0}^p \\
&\leq \varepsilon.
\end{aligned}$$

For the case $p = 2$, we proceed in the same way using isometry inequality to obtain

$$\begin{aligned}
&\mathbf{E}\big\|\Psi X(t+\tau) - \Psi X(t)\big\|^2 \\
&= \int_{-\infty}^t \mathbf{E}\big\|T(t-s)[G(s+\tau, X(s+\tau)) - G(s, X(s))]\big\|^2 \, ds \\
&\leq M^2 \int_{-\infty}^t e^{-2\delta(t-s)} \mathbf{E}\big\|G(s+\tau, X(s+\tau)) - G(s, X(s))\big\|_{\mathbb{L}_2^0}^2 \, ds \\
&\leq M^2 \Big(\int_{-\infty}^t e^{-2\delta(t-s)} \, ds\Big) \sup_{s\in\mathbb{R}} \mathbf{E}\big\|G(s+\tau, X(s+\tau)) - G(s, X(s))\big\|_{\mathbb{L}_2^0}^2 \\
&< \varepsilon.
\end{aligned}$$

Hence, $\Psi X(\cdot)$ is p-th mean almost periodic.

To complete the proof, we will show that Λ is a contraction. For that, let X and Y be in $AP(\mathbb{R}; L^p(\Omega, \mathbb{H}))$. Proceeding as before starting with the case where $p > 2$, we obtain

$$
\begin{aligned}
&\mathbf{E}\big\|\Lambda X(t)-\Lambda Y(t)\big\|^p \\
&\le 2^{p-1}\mathbf{E}\left\|\int_{-\infty}^t T(t-\sigma)\big[F(\sigma,X(\sigma))-F(\sigma,Y(\sigma))\big]\,d\sigma\right\|^p \\
&\quad+2^{p-1}\mathbf{E}\left\|\int_{-\infty}^t T(t-\sigma)\big[G(\sigma,X(\sigma))-G(\sigma,Y(\sigma))\big]\,d\mathbb{W}(\sigma)\right\|^p .
\end{aligned}
$$

Using assumption $(3H)_0$, an application of Hölder's inequality, Proposition 3.23, followed by $(5H)_1$ and $(5H)_2$, gives

$$
\begin{aligned}
&\mathbf{E}\big\|\Lambda X(t)-\Lambda Y(t)\big\|^p \\
&\le 2^{p-1}\mathbf{E}\Big[\int_{-\infty}^t \big\|T(t-\sigma)\big\|\big\|F(\sigma,X(\sigma))-F(\sigma,Y(\sigma))\big\|\,d\sigma\Big]^p \\
&+2^{p-1}C_p\mathbf{E}\Big[\int_{-\infty}^t \big\|T(t-\sigma)\big\|^2\,\big\|G(\sigma,X(\sigma))-G(\sigma,Y(\sigma))\big\|^2_{\mathbb{L}_2^0}\,d\sigma\Big]^{p/2} \\
&\le 2^{p-1}\left(\int_{-\infty}^t e^{-\delta(t-s)}\,d\sigma\right)^{p-1}\times \\
&\times\left(\int_{-\infty}^t e^{-\delta(t-s)}\mathbf{E}\big\|F(\sigma,X(\sigma))-F(\sigma,Y(\sigma))\big\|^p\,d\sigma\right) \\
&+2^{p-1}C_p\left(\int_{-\infty}^t e^{-\frac{p}{p-2}\,\delta(t-s)}\,d\sigma\right)^{\frac{p-2}{2}}\times \\
&\times\left(\int_{\infty}^t e^{-\frac{p}{2}\,\delta(t-s)}\,\mathbf{E}\big\|G(\sigma,X(\sigma))-G(\sigma,Y(\sigma))\big\|^p_{\mathbb{L}_2^0}\,d\sigma\right) \\
&\le 2^{p-1}M^p K_F\left(\int_{-\infty}^t e^{-\delta\,(t-\sigma)}\,d\sigma\right)^p\big\|X-Y\big\|_\infty^p \\
&+2^{p-1}C_p M^p K_G\left(\int_{-\infty}^t e^{-\frac{p\,\delta}{p-2}(t-\sigma)}\,d\sigma\right)^{\frac{p-2}{2}}\left(\int_{\infty}^t e^{-\frac{p\,\delta}{2}(t-\sigma)}\,d\sigma\right)\big\|X-Y\big\|_\infty^p \\
&\le 2^p M^p\left[K_F\left(\frac{1}{\delta^p}\right)+C_p K_G\left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{2}}\left(\frac{1}{\delta\,p}\right)\right]\big\|X-Y\big\|_\infty^p.
\end{aligned}
$$

As to the case $p=2$, we have

$$\begin{aligned}
&\mathbf{E}\big\|\Lambda X(t)-\Lambda Y(t)\big\|^2\\
&\le 2M^2\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\,d\sigma\Big)\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\,\mathbf{E}\big\|f(\sigma,X(\sigma))-f(\sigma,Y(\sigma))\big\|^2\,d\sigma\Big)\\
&\quad +2M^2\int_{-\infty}^{t} e^{-2\delta(t-\sigma)}\mathbf{E}\big\|g(\sigma,X(\sigma))-g(\sigma,Y(\sigma))\big\|^2\,d\sigma\\
&\le 2M^2\cdot C_1\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\,d\sigma\Big)\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\,\mathbf{E}\big\|X(\sigma)-Y(\sigma)\big\|^2\,d\sigma\Big)\\
&\quad +2M^2\cdot C_2\int_{-\infty}^{t} e^{-2\delta(t-\sigma)}\mathbf{E}\big\|X(\sigma)-Y(\sigma)\big\|^2\,d\sigma\\
&\le 2M^2\cdot C_1\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\,d\sigma\Big)^2\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|X(\sigma)-Y(\sigma)\big\|^2\\
&\quad +2M^2\cdot C_2\Big(\int_{-\infty}^{t} e^{-2\delta(t-\sigma)}\,d\sigma\Big)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|X(\sigma)-Y(\sigma)\big\|^2\\
&\le 2M^2\Big(\frac{C_1}{\delta^2}+\frac{C_2}{\delta}\Big)\big\|X-Y\big\|_\infty^2.
\end{aligned}$$

Consequently, if $\Theta_p < 1$, then Λ is a contraction. The use of the Banach fixed-point principle completes the proof.

5.2 The Nonautonomous Case

5.2.1 Introduction

Let $(\mathbb{H},\|\cdot\|)$ be a real (separable) Hilbert space. This section is mainly concerned with the existence of p-th mean almost periodic solutions to nonautonomous semilinear stochastic differential equations

$$dX(t) = A(t)X(t)\,dt + F(t,X(t))\,dt + G(t,X(t))\,d\mathbb{W}(t),\ t\in\mathbb{R}, \tag{5.3}$$

where $(A(t))_{t\in\mathbb{R}}$ is a family of densely defined closed linear operators satisfying "Acquistapace-Terreni" conditions (2.38) and (2.39), $F:\mathbb{R}\times L^p(\Omega,\mathbb{H})\to L^p(\Omega,\mathbb{H})$ and $G:\mathbb{R}\times L^p(\Omega,\mathbb{H})\to L^p(\Omega,\mathbb{L}_2^0)$ are jointly continuous satisfying some additional conditions, and $\mathbb{W}(t)$ is a $\mathscr{Q}$-Wiener process with values in $\mathbb{K}$.

The existence of almost periodic (respectively, periodic) solutions to autonomous stochastic differential equations has been studied by many authors, see, e.g., [3] and [20]. In Da Prato and Tudor [46], the existence of almost periodic solutions to (5.3) in the case when $A(t)$ is periodic, that is, $A(t+T)=A(t)$ for each $t\in\mathbb{R}$ for some $T>0$, was established. In this section, it goes back to studying the existence and uniqueness of a square-mean almost periodic solution to (5.3) when the operators $A(t)$ satisfy "Acquistapace–Terreni" conditions (2.38) and (2.39).

Here we assume that $A(t) : D(A(t)) \subset L^p(\Omega;\mathbb{H}) \to L^p(\Omega;\mathbb{H})$ is a family of densely defined closed linear operators on a common domain $\mathscr{D} = D(A(t))$, which is independent of t and dense in $L^p(\Omega;\mathbb{H})$, and $F : \mathbb{R} \times L^p(\Omega;\mathbb{H}) \to L^p(\Omega;\mathbb{H})$ and $G : \mathbb{R} \times L^p(\Omega;\mathbb{H}) \to L^p(\Omega;\mathbb{L}_2^0)$ are jointly continuous functions.

We suppose that the system

$$\begin{cases} u'(t) = A(t)u(t) \quad t \geq s, \\ \\ u(s) = x \in L^p(\Omega;\mathbb{H}), \end{cases} \tag{5.4}$$

has an associated evolution family of operators $\{U(t,s) : t \geq s \text{ with } t,s \in \mathbb{R}\}$, which is uniformly asymptotically stable.

5.2.2 Existence of p-th Mean Almost Periodic Solutions

Throughout this subsection, we require the following assumption in addition to $(5H)_1$ and $(5H)_2$ from the previous section:

$(5H)_3$ The operators $A(t)$, $U(r,s)$ commute and that the evolution family $U(t,s)$ is asymptotically stable. Namely, there exist some constants $M, \delta > 0$ such that

$$\|U(t,s)\| \leq Me^{-\delta(t-s)} \text{ for every } t \geq s.$$

In addition, $R(\lambda_0, A(\cdot)) \in AP(\mathbb{R}; B(L^p(\Omega,\mathbb{H})))$ for λ_0 in Eq. (2.38).

In order to study Eq. (5.3) we need the following lemma which is an immediate consequence of [136, Proposition 4.4].

Lemma 5.1. *Suppose $A(t)$ satisfies the "Acquistapace–Terreni" conditions, $U(t,s)$ is exponentially stable, and $R(\lambda_0, A(\cdot)) \in AP(\mathbb{R}; B(L^p(\Omega,\mathbb{H})))$. Let $h > 0$. Then, for any $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains at least a number τ with the property that*

$$\left\|U(t+\tau, s+\tau) - U(t,s)\right\| \leq \varepsilon\, e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \geq h$.

Definition 5.2. An $\mathscr{F}_t$-progressively measurable process $\{X(t)\}_{t\in\mathbb{R}}$ is called a mild solution of Eq. (5.3) on $\mathbb{R}$ if

$$\begin{aligned} X(t) = U(t,s)X(s) &+ \int_s^t U(t,\sigma)F(\sigma, X(\sigma))\, d\sigma \\ &+ \int_s^t U(t,\sigma)G(\sigma, X(\sigma))\, d\mathbb{W}(\sigma) \end{aligned} \tag{5.5}$$

for all $t \geq s$ for each $s \in \mathbb{R}$.

The main result of this section can be formulated as follows:

Theorem 5.2. *Under assumptions* $(5H)_1$, $(5H)_2$, *and* $(5H)_3$, *then Eq.* (5.3) *has a unique p-th mean almost periodic mild solution, which can be explicitly expressed as follows:*

$$X(t) = \int_{-\infty}^{t} U(t,\sigma)F(\sigma,X(\sigma))\,d\sigma + \int_{-\infty}^{t} U(t,\sigma)G(\sigma,X(\sigma))\,d\mathbb{W}(\sigma) \text{ for each } t \in \mathbb{R}$$

whenever

$$\Theta := 2^p M^p \left[K_F \left(\frac{1}{\delta^p}\right) + C_p K_G \left(\frac{p-2}{p\,\delta}\right)^{\frac{p-2}{2}} \left(\frac{1}{p\,\delta}\right)\right]$$

for $p > 2$ *and*

$$\Theta := M^2 \left(2\frac{K}{\delta^2} + \frac{K'}{\delta}\right) < 1$$

for $p = 2$.

Proof. First of all, note that

$$\begin{aligned} X(t) = & \int_{-\infty}^{t} U(t,\sigma)F(\sigma,X(\sigma))\,d\sigma \\ & + \int_{-\infty}^{t} U(t,\sigma)G(\sigma,X(\sigma))\,d\mathbb{W}(\sigma) \end{aligned} \tag{5.6}$$

is well-defined and satisfies

$$\begin{aligned} X(t) = & \ U(t,s)X(s) + \int_{s}^{t} U(t,\sigma)F(\sigma,X(\sigma))\,d\sigma \\ & + \int_{s}^{t} U(t,\sigma)G(\sigma,X(\sigma))\,d\mathbb{W}(\sigma) \end{aligned}$$

for all $t \geq s$ for each $s \in \mathbb{R}$, and hence X given by (5.6) is a mild solution to (5.3).

Define $\Lambda X(t) = \Phi X(t) + \Psi X(t)$, where

$$\Phi X(t) := \int_{-\infty}^{t} U(t,\sigma)\varphi X(\sigma)\,d\sigma,$$

and

$$\Psi X(t) := \int_{-\infty}^{t} U(t,\sigma)\psi X(\sigma)\,d\mathbb{W}(\sigma),$$

with $\varphi X(t) = F(t,X(t))$ and $\psi X(t) = G(t,X(t))$.

Let us first show that $\Phi X(\cdot)$ and $\Psi X(\cdot)$ are p-th mean almost periodic whenever X is. Indeed, assuming that X is p-th mean almost periodic and using assumption $(5H)_1$, Theorem 4.4, and Lemma 5.1, given $\varepsilon > 0$, one can find $l_\varepsilon > 0$ such that any interval of length l_ε contains at least τ with the property that

$$\left\| U(t+\tau, s+\tau) - U(t,s) \right\| \leq \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \geq \varepsilon$, and

$$\mathbf{E}\left\| \varphi X(\sigma+\tau) - \varphi X(\sigma) \right\|^p < \eta$$

for each $\sigma \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Moreover, it follows from Lemma 4.1 (ii) that there exists a positive constant K_1 such that

$$\sup_{\sigma \in \mathbb{R}} \mathbf{E}\left\| \varphi X(\sigma) \right\|^p \leq K_1 \,.$$

Now, using assumption $(3H)_0$ and Hölder's inequality, we obtain

$$\begin{aligned}
&\mathbf{E}\left\| \Phi X(t+\tau) - \Phi X(t) \right\|^p \\
&\leq 3^{p-1}\, \mathbf{E}\Big[\int_0^\infty \left\| U(t+\tau, t+\tau-s) \right\| \left\| \varphi X(t+\tau-s) - \varphi X(t-s) \right\| ds \Big]^p \\
&+ 3^{p-1}\, \mathbf{E}\Big[\int_\varepsilon^\infty \left\| U(t+\tau, t+\tau-s) - U(t,t-s) \right\| \left\| \varphi X(t-s) \right\| ds \Big]^p \\
&+ 3^{p-1}\, \mathbf{E}\Big[\int_0^\varepsilon \left\| U(t+\tau, t+\tau-s) - U(t,t-s) \right\| \left\| \varphi X(t-s) \right\| ds \Big]^p \\
&\leq 3^{p-1} M^p\, \mathbf{E}\Big[\int_0^\infty e^{-\delta s} \left\| \varphi X(t+\tau-s) - \varphi X(t-s) \right\| ds \Big]^p \\
&+ 3^{p-1} \varepsilon^p\, \mathbf{E}\Big[\int_\varepsilon^\infty e^{-\frac{\delta}{2} s} \left\| \varphi X(t-s) \right\| ds \Big]^p + 3^{p-1} M^p\, \mathbf{E}\Big[\int_0^\varepsilon 2e^{-\delta s} \left\| \varphi X(t-s) \right\| ds \Big]^p \\
&\leq 3^{p-1} M^p \Big(\int_0^\infty e^{-\delta s}\, ds \Big)^{p-1} \Big(\int_0^\infty e^{-\delta s}\, \mathbf{E}\left\| \varphi X(t+\tau-s) - \varphi X(t-s) \right\|^p ds \Big) \\
&+ 3^{p-1} \varepsilon^p \Big(\int_0^\infty e^{-\delta s}\, ds \Big)^{p-1} \Big(\int_\varepsilon^\infty e^{-\frac{\delta p s}{2}}\, \mathbf{E}\left\| \varphi X(t-s) \right\|^p ds \Big) \\
&+ 6^{p-1} M^p \Big(\int_0^\varepsilon e^{-\delta s}\, ds \Big)^{p-1} \Big(\int_0^\varepsilon e^{-\frac{\delta p s}{2}}\, \mathbf{E}\left\| \varphi X(t-s) \right\|^p ds \Big) \\
&\leq 3^{p-1} M^p \Big(\int_0^\infty e^{-\delta s}\, ds \Big)^p \sup_{s \in \mathbb{R}} \mathbf{E}\left\| \varphi X(t+\tau-s) - \varphi X(t-s) \right\|^p \\
&+ 3^{p-1} \varepsilon^p \Big(\int_\varepsilon^\infty e^{-\delta s}\, ds \Big)^p \sup_{s \in \mathbb{R}} \mathbf{E}\left\| \varphi X(t-s) \right\|^p \\
&+ 6^{p-1} M^p \Big(\int_0^\varepsilon e^{-\delta s}\, ds \Big)^p \sup_{s \in \mathbb{R}} \mathbf{E}\left\| \varphi X(t-s) \right\|^p \\
&\leq 3^{p-1} M^p \Big(\frac{1}{\delta^p} \Big) \eta + 3^{p-1} M^p K_1 \Big(\frac{1}{\delta^p} \Big) \varepsilon^p + 6^{p-1} M^p \varepsilon^p K_1 \varepsilon^p.
\end{aligned}$$

As to $\Psi X(\cdot)$, we again split the proof in two cases: $p > 2$ and $p = 2$. Let us start with the case where $p > 2$. Assuming that X is p-th mean almost periodic and using assumption $(5H)_2$, Theorem 4.4, and Lemma 5.1, given $\varepsilon > 0$, one can find $l_\varepsilon > 0$ such that any interval of length l_ε contains at least τ with the property that

$$\big\|U(t+\tau, s+\tau) - U(t,s)\big\| \leq \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \geq \varepsilon$, and

$$\mathbf{E}\big\|\psi X(\sigma+\tau) - \psi X(\sigma)\big\|^p < \eta$$

for each $\sigma \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Moreover, it follows from Lemma 4.1 (ii) that there exists a positive constant K_2 such that

$$\sup_{\sigma\in\mathbb{R}} \mathbf{E}\big\|\psi X(\sigma)\big\|^p \leq K_2\,.$$

Now

$$\begin{aligned}
&\mathbf{E}\big\|\Psi X(t+\tau) - \Psi X(t)\big\|^p \\
&\leq 3^{p-1}\,\mathbf{E}\left\|\int_0^\infty U(t+\tau, t+\tau-s)\left[\psi X(t+\tau-s) - \psi X(t-s)\right] d\mathbb{W}(s)\right\|^p \\
&+3^{p-1}\,\mathbf{E}\left\|\int_\varepsilon^\infty \left[U(t+\tau, t+\tau-s) - U(t,t-s)\right] \psi X(t-s)\, d\mathbb{W}(s)\right\|^p \\
&+3^{p-1}\,\mathbf{E}\left\|\int_0^\varepsilon \left[U(t+\tau, t+\tau-s) - U(t,t-s)\right] \psi X(t-s)\, d\mathbb{W}(s)\right\|^p.
\end{aligned}$$

The next step consists in proving the p-th mean almost periodicity of $\Psi X(\cdot)$. Using assumption $(3H)_0$, Hölder's inequality, and Proposition 3.23, we have

$$\begin{aligned}
&\mathbf{E}\big\|\Psi X(t+\tau) - \Psi X(t)\big\|^p \\
&\leq 3^{p-1}\,C_p\,\mathbf{E}\Big[\int_0^\infty \big\|U(t+\tau, t+\tau-s)\big\|^2 \big\|\psi X(t+\tau-s) - \psi X(t-s)\big\|^2_{\mathbb{L}_2^0} ds\Big]^{p/2} \\
&+3^{p-1}\,C_p\,\mathbf{E}\Big[\int_\varepsilon^\infty \big\|U(t+\tau, t+\tau-s) - U(t,t-s)\big\|^2 \big\|\psi X(t-s)\big\|^2_{\mathbb{L}_2^0} ds\Big]^{p/2} \\
&+3^{p-1}\,C_p\,\mathbf{E}\Big[\int_0^\varepsilon \big\|U(t+\tau, t+\tau-s) - U(t,t-s)\big\|^2 \big\|\psi X(t-s)\big\|^2_{\mathbb{L}_2^0} ds\Big]^{p/2} \\
&\leq 3^{p-1}\,C_p\,M^p\,\mathbf{E}\Big[\int_0^\infty e^{-2\delta s} \big\|\psi X(t+\tau-s) - \psi X(t-s)\big\|^2_{\mathbb{L}_2^0} ds\Big]^{p/2} \\
&+3^{p-1}\,C_p\,\varepsilon^p\mathbf{E}\Big[\int_\varepsilon^\infty e^{-\delta s} \big\|\psi X(t-s)\big\|^2_{\mathbb{L}_2^0} ds\Big]^{p/2}
\end{aligned}$$

$$
\begin{aligned}
&+3^{p-1}\,2^{p/2}\,C_p\,\mathbf{E}\Big[\int_0^{\varepsilon} e^{-2\delta s}\big\|\psi X(t-s)\big\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}\\
&\le 3^{p-1}\,C_p\,M^p\Big(\int_0^{\infty} e^{-\frac{p\delta s}{p-2}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_0^{\infty} e^{-\frac{p\delta s}{2}}\,\big\|\psi X(t+\tau-s)-\psi X(t-s)\big\|^p_{\mathbb{L}_2^0}\,ds\Big)\\
&+3^{p-1}\,C_p\,\varepsilon^p\Big(\int_{\varepsilon}^{\infty} e^{-\frac{p\delta s}{2(p-2)}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_{\varepsilon}^{\infty} e^{-\frac{p\delta s}{4}}\,\mathbf{E}\big\|\psi X(t-s)\big\|^p_{\mathbb{L}_2^0}\,ds\Big)\\
&+3^{p-1}\,2^{p/2}\,C_p\,M^p\Big(\int_0^{\varepsilon} e^{-\frac{p\delta s}{p-2}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_0^{\varepsilon} e^{-\frac{p\delta s}{2}}\,\mathbf{E}\big\|\psi X(t-s)\big\|^p_{\mathbb{L}_2^0}\,ds\Big)\\
&\le 3^{p-1}\,C_p\,M^p\,\eta\Big(\int_0^{\infty} e^{-\frac{p\delta s}{p-2}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_0^{\infty} e^{-\frac{p\delta s}{2}}\,ds\Big)\\
&+3^{p-1}\,C_p\,\varepsilon^p\,K_2\Big(\int_{\varepsilon}^{\infty} e^{-\frac{p\delta s}{2(p-2)}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_{\varepsilon}^{\infty} e^{-\frac{p\delta s}{4}}\,ds\Big)\\
&+3^{p-1}\,2^{p/2}\,C_p\,M^p\,K_2\Big(\int_0^{\varepsilon} e^{-\frac{p\delta s}{p-2}}\,ds\Big)^{\frac{p-2}{2}}\Big(\int_0^{\varepsilon} e^{-\frac{p\delta s}{2}}\,ds\Big)\\
&\le 3^{p-1}\,C_p\,M^p\,\eta\Big(\frac{p-2}{p\,\delta}\Big)^{p-2}\Big(\frac{2}{p\,\delta}\Big)\\
&+3^{p-1}\,C_p\,\varepsilon^p\,K_2\Big(\frac{2(p-2)}{p\,\delta}\Big)^{\frac{p-2}{2}}\Big(\frac{4}{p\,\delta}\Big)+3^{p-1}\,2^{p/2}\,C_p\,M^p\,K_2\varepsilon^p.
\end{aligned}
$$

As to the case $p=2$, we proceed in the same way using isometry inequality to obtain

$$
\begin{aligned}
&\mathbf{E}\big\|(\Psi X)(t+\tau)-(\Psi X)(t)\big\|^2\\
&\le 3\,M^2\Big(\int_0^{\infty} e^{-2\delta s}\,ds\Big)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|\psi X(\sigma+\tau)-\psi X(\sigma)\big\|^2_{\mathbb{L}_2^0}\\
&+3\,\varepsilon^2\Big(\int_{\varepsilon}^{\infty} e^{-\delta s}\,ds\Big)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|\psi X(\sigma)\big\|^2_{\mathbb{L}_2^0}+6\,M^2\Big(\int_0^{\varepsilon} e^{-2\delta s}\,ds\Big)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|\psi X(\sigma)\big\|^2_{\mathbb{L}_2^0}\\
&\le 3\,\Big[\eta\,\frac{M^2}{2\delta}+\varepsilon\frac{K_2}{\delta}+2\,\varepsilon\,K_2\Big].
\end{aligned}
$$

Hence, $\Psi X(\cdot)$ is p-th mean almost periodic.

Finally, we will show that Λ is a contraction. Let X and Y be in $AP(\mathbb{R};L^p(\Omega,\mathbb{H}))$. Proceeding as before starting with the case where $p>2$, we obtain

$$
\begin{aligned}
&\mathbf{E}\big\|\Lambda X(t)-\Lambda Y(t)\big\|^p \\
&\leq 2^{p-1}\mathbf{E}\Big\|\int_{-\infty}^{t} U(t,\sigma)\big[\varphi X(\sigma)-\varphi Y(\sigma)\big]\,d\sigma\Big\|^p \\
&\quad +2^{p-1}\mathbf{E}\Big\|\int_{-\infty}^{t} U(t,\sigma)\big[\psi X(\sigma)-\psi Y(\sigma)\big]\,d\mathbb{W}(\sigma)\Big\|^p .
\end{aligned}
$$

Using assumption $(5H)_3$, an application of Hölder's inequality, Proposition 3.23, followed by $(5H)_1$ and $(5H)_2$, gives

$$
\begin{aligned}
&\mathbf{E}\big\|\Lambda X(t)-\Lambda Y(t)\big\|^p \\
&\leq 2^{p-1}\mathbf{E}\Big[\int_{-\infty}^{t}\big\|U(t,\sigma)\big\|\,\big\|\varphi X(\sigma)-\varphi Y(\sigma)\big\|\,d\sigma\Big]^p \\
&\quad +2^{p-1}C_p\mathbf{E}\Big[\int_{-\infty}^{t}\big\|U(t,\sigma)\big\|^2\,\big\|\psi X(\sigma)-\psi Y(\sigma)\big\|^2_{\mathbb{L}_2^0}\,d\sigma\Big]^{p/2} \\
&\leq 2^{p-1}\,M^p\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\Big)^{p-1}\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\,\mathbf{E}\big\|\varphi X(\sigma)-\varphi Y(\sigma)\big\|^p\,d\sigma\Big) \\
&\quad +2^{p-1}\,C_p\Big(\int_{-\infty}^{t} e^{-\frac{p}{p-2}\,\delta(t-s)}\,d\sigma\Big)^{\frac{p-2}{2}}\Big(\int_{-\infty}^{t} e^{-\frac{p}{2}\,\delta(t-s)}\,\mathbf{E}\big\|\psi X(\sigma)-\psi Y(\sigma)\big\|^p_{\mathbb{L}_2^0}\,d\sigma\Big) \\
&\leq 2^{p-1}\,M^p\,K_F\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\,d\sigma\Big)^p\,\big\|X-Y\big\|^p_\infty \\
&\quad +2^{p-1}\,C_p\,M^p\,K_G\Big(\int_{-\infty}^{t} e^{-\frac{p\,\delta}{p-2}(t-\sigma)}\,d\sigma\Big)^{\frac{p-2}{2}}\Big(\int_{-\infty}^{t} e^{-\frac{p\,\delta}{2}(t-\sigma)}\,d\sigma\Big)\,\big\|X-Y\big\|^p_\infty \\
&= 2^p\,M^p\,\Big[K_F\Big(\frac{1}{\delta^p}\Big)+C_p\,K_G\Big(\frac{p-2}{p\,\delta}\Big)^{\frac{p-2}{2}}\Big(\frac{1}{p\,\delta}\Big)\Big]\,\big\|X-Y\big\|^p_\infty=\Theta\cdot\big\|X-Y\big\|^p_\infty.
\end{aligned}
$$

As to the case $p=2$, we have

$$\begin{aligned}
&\mathbf{E}\big\|\Lambda X(t)-\Lambda Y(t)\big\|^2\\
&\le 2M^2\Big(\int_{-\infty}^t e^{-\delta(t-s)}\,ds\Big)\Big(\int_{-\infty}^t e^{-\delta(t-s)}\,\mathbf{E}\big\|\varphi X(s)-\varphi Y(s)\big\|^2\,ds\Big)\\
&\quad +2M^2\int_{-\infty}^t e^{-2\delta(t-s)}\mathbf{E}\big\|\psi X(s)-\psi Y(s)\big\|^2_{\mathbb{L}_2^0}\,ds\\
&\le 2M^2\cdot K\Big(\int_{-\infty}^t e^{-\delta(t-s)}\,ds\Big)\Big(\int_{-\infty}^t e^{-\delta(t-s)}\,\mathbf{E}\big\|X(s)-Y(s)\big\|^2\,ds\Big)\\
&\quad +2M^2\cdot K'\int_{-\infty}^t e^{-2\delta(t-s)}\mathbf{E}\big\|X(s)-Y(s)\big\|^2\,ds\\
&\le 2M^2\cdot K\Big(\int_{-\infty}^t e^{-\delta(t-s)}\,ds\Big)^2\sup_{s\in\mathbb{R}}\mathbf{E}\big\|X(s)-Y(s)\big\|^2\Big)\\
&\quad +2M^2\cdot K'\Big(\int_{-\infty}^t e^{-2\delta(t-s)}\,ds\Big)\sup_{s\in\mathbb{R}}\mathbf{E}\big\|X(s)-Y(s)\big\|^2\\
&\le 2M^2\Big(\frac{K}{\delta^2}+\frac{K'}{\delta}\Big)\big\|X-Y\big\|^2_\infty\\
&\le \Theta\cdot\big\|X-Y\big\|^2_\infty .
\end{aligned}$$

Consequently, if $\Theta<1$, then Λ is a contraction mapping. One completes the proof by the Banach fixed-point principle.

5.2.3 Example

Let $\mathscr{O}\subset\mathbb{R}^n$ be a bounded subset whose boundary $\partial\mathscr{O}$ is of class C^2 and being locally on one side of $\mathscr{O}$.

Consider the parabolic stochastic partial differential equation

$$d_tX(t,\xi)=\{A(t,\xi)X(t,\xi)+F(t,X(t,\xi))\}\,d_t+G(t,X(t,\xi))\,d\mathbb{W}(t), \tag{5.7}$$

$$\sum_{i,j=1}^n n_i(\xi)a_{ij}(t,\xi)d_iX(t,\xi)=0,\ \ t\in\mathbb{R},\ \ \xi\in\partial\mathscr{O}, \tag{5.8}$$

where $d_t=\dfrac{d}{dt}$, $d_i=\dfrac{d}{d\xi_i}$, $n(\xi)$ is the outer unit normal vector, the family of operators $A(t,\xi)$ are formally given by

$$A(t,\xi)=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(t,\xi)\frac{\partial}{\partial x_j}\right)+c(t,\xi),\quad t\in\mathbb{R},\ \ \xi\in\mathscr{O},$$

W is a real-valued Brownian motion, and a_{ij}, c $(i,j=1,2,\dots,n)$ satisfy the following conditions:
$(5H)_{3'}$

(i) The coefficients $(a_{ij})_{i,j=1,\dots,n}$ are symmetric, that is, $a_{ij}=a_{ji}$ for all $i,j=1,\dots,n$. Moreover, $a_{ij}\in C_b^{\mu}(\mathbb{R},L^2(\Omega,C(\mathscr{O})))\cap \overline{BC}(\mathbb{R},L^2(\Omega,C^1(\mathscr{O})))\cap A\overline{P}(\mathbb{R};L^2(\Omega,L^2(\mathscr{O})))$ for all $i,j=1,\dots,n$, and

$$c\in C_b^{\mu}(\mathbb{R},L^2(\Omega,L^2(\mathscr{O})))\cap BC(\mathbb{R},L^2(\Omega,C(\overline{\mathscr{O}})))\cap AP(\mathbb{R};L^2(\Omega,L^1(\mathscr{O})))$$

for some $\mu\in(1/2,1]$.

(ii) There exists $\varepsilon_0>0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t,\xi)\eta_i\eta_j \geq \varepsilon_0|\eta|^2,$$

for all $(t,\xi)\in\mathbb{R}\times\overline{\mathscr{O}}$ and $\eta\in\mathbb{R}^n$.

Under previous assumptions, the existence of an evolution family $U(t,s)$ satisfying $(5H)_3$ is obtained, see, e.g., [136].

Set $\mathbb{H}=L^2(\mathscr{O})$. For each $t\in\mathbb{R}$ define an operator $A(t)$ on $L^2(\Omega;\mathbb{H})$ by

$$\mathscr{D}(A(t))=\left\{X\in L^2(\Omega,H^2(\mathscr{O})) : \sum_{i,j=1}^{n} n_i(\cdot)a_{ij}(t,\cdot)d_iX(t,\cdot)=0 \ \text{ on } \ \partial\mathscr{O}\right\} \ \text{ and}$$

$$A(t)X=A(t,\xi)X(\xi), \ \text{ for all } \ X\in\mathscr{D}(A(t)) \,.$$

Therefore, under assumptions $(5H)_1$, $(5H)_2$, $(5H)_3$, and $(5H)_{3'}$, then Eqs.(5.7)–(5.8) have a unique mild solution, which obviously is square-mean almost periodic, whenever M is small enough.

5.3 Bibliographical Notes

All the main results presented in this chapter are based on some recent work by the authors, see, e.g., [20, 21].

Chapter 6
Existence Results for Some Partial Stochastic Differential Equations

This chapter is devoted to the study of the existence of solutions to some partial stochastic differential equations inspired by their deterministic counterparts [60, 52, 54, 91, 92, 94, 95, 50, 97, 98, 99]. Applications arise in control systems, for instance.

In this chapter, $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ denotes a real Hilbert space which is separable and $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$ is a filtered probability space. If $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{H} \to \mathbb{H}$ is a linear operator, we then define the corresponding operator $A : D(A) \subset L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{H})$ as follows: $X \in D(A)$ and $AX = Y$ if and only if $X, Y \in L^p(\Omega, \mathbb{H})$ and $\mathscr{A}X(\omega) = Y(\omega)$ for all $\omega \in \Omega$.

6.1 The Autonomous Case

Let $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{H} \to \mathbb{H}$ be a sectorial linear operator. For $\alpha \in (0,1)$, let $\mathbb{H}_\alpha$ denote the intermediate Banach space between $D(\mathscr{A})$ and $\mathbb{H}$ as seen in Chapter 2.

The present section is inspired by the previous chapter and consists of studying the existence of p-th mean almost periodic solutions to the stochastic differential equation of the form

$$d\Big(X(\omega,t) + f(t, \mathscr{B}X(\omega,t))\Big) = \Big[\mathscr{A}X(\omega,t) + g(t, \mathscr{C}X(\omega,t))\Big]dt \qquad (6.1)$$
$$+h(t, \mathscr{L}X(\omega,t))d\mathbb{W}(\omega,t)$$

for all $t \in \mathbb{R}$ and $\omega \in \Omega$, where $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{H} \to \mathbb{H}$ is a sectorial linear operator whose corresponding analytic semigroup is hyperbolic, that is, $\sigma(\mathscr{A}) \cap i\mathbb{R} = \emptyset$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{L}$ are (possibly unbounded) linear operators on $\mathbb{H}$) and $f : \mathbb{R} \times \mathbb{H} \to \mathbb{H}_\beta$ $(0 < \alpha < \frac{1}{p} < \beta < 1)$, $g : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$, and $h : \mathbb{R} \times \mathbb{H} \to \mathbb{L}_2^0$ are jointly continuous functions.

To analyze Eq. (6.1), our strategy consists in studying the existence of p-th mean almost periodic solutions to the corresponding class of stochastic differential equations of the form

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$$d\Big(X(t)+F(t,BX(t))\Big) = \Big[AX(t)+G(t,CX(t))\Big]dt \tag{6.2}$$

$$+H(t,LX(t))d\mathbb{W}(t) \tag{6.3}$$

for all $t \in \mathbb{R}$, where $A : D(A) \subset L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{H})$ is a sectorial linear operator whose corresponding analytic semigroup is hyperbolic, that is, $\sigma(A) \cap i\mathbb{R} = \emptyset$, B, C, and L are (possibly unbounded) linear operators on $L^p(\Omega,\mathbb{H})$ and $F : \mathbb{R} \times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{H}_\beta)$ $(0 < \alpha < \frac{1}{p} < \beta < 1)$, $G : \mathbb{R} \times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{H})$, and $H : \mathbb{R} \times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{L}_2^0)$ are jointly continuous functions satisfying some additional assumptions, and $\mathbb{W}$ is a $\mathscr{Q}$-Wiener process with values in $\mathbb{K}$.

Although the existence and uniqueness of p-th mean almost periodic solutions to Eq. (6.1) in the case when A is sectorial is an important topic with some interesting applications, it is still an untreated question and constitutes the main motivation of this problem. The techniques we use to derive sufficient conditions for the existence and uniqueness of a p-th mean almost periodic solution to (6.1) are based on the method of analytic semigroups associated with sectorial operators and the Banach fixed-point principle.

6.1.1 Existence of p-th Mean Almost Periodic Solutions

Definition 6.1. Let $\alpha \in (0,1)$. A continuous random function, $X : \mathbb{R} \to L^p(\Omega;\mathbb{H}_\alpha)$ is said to be a mild solution of Eq. (6.1) provided that the function
$s \to \mathbf{E}\Big\|AT(t-s)PF(s,BX(s))\Big\|^p$ is integrable on $(-\infty,t)$, the function
$s \to \mathbf{E}\Big\|AT(t-s)QF(s,BX(s))\Big\|^p$ is integrable on (t,∞) for each $t \in \mathbb{R}$, and

$$\begin{aligned} X(t) = -F(t,BX(t)) - \int_{-\infty}^{t} AT(t-s)PF(s,BX(s))\,ds + \int_{t}^{\infty} AT(t-s)QF(s,BX(s))\,ds \\ + \int_{-\infty}^{t} T(t-s)P\,G(s,CX(s))\,ds - \int_{t}^{\infty} T(t-s)Q\,G(s,CX(s))\,ds \\ + \int_{-\infty}^{t} T(t-s)P\,H(s,LX(s))\,d\mathbb{W}(s) - \int_{t}^{\infty} T(t-s)Q\,H(s,LX(s))\,d\mathbb{W}(s) \end{aligned}$$

for each $t \in \mathbb{R}$.

Here and below, we let $Q = I - P$ for a projection P.

Define Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, and Γ_6 respectively by the nonlinear integral operators:

$$(\Gamma_1 X)(t) := \int_{-\infty}^{t} AT(t-s)PF(s,BX(s))\,ds,$$
$$(\Gamma_2 X)(t) := \int_{t}^{\infty} AT(t-s)QF(s,BX(s))\,ds,$$
$$(\Gamma_3 X)(t) := \int_{-\infty}^{t} T(t-s)PG(s,CX(s))\,ds,$$
$$(\Gamma_4 X)(t) := \int_{t}^{\infty} T(t-s)QG(s,CX(s))\,ds,$$
$$(\Gamma_5 X)(t) := \int_{-\infty}^{t} T(t-s)PH(s,LX(s))\,d\mathbb{W}(s), \text{ and}$$
$$(\Gamma_6 X)(t) := \int_{t}^{\infty} T(t-s)QH(s,LX(s))\,d\mathbb{W}(s).$$

To discuss the existence of p-th mean almost periodic solution to Eq. (6.1) we need to set some assumptions on A, B, C, L, F, G, and H. First of all, note that for $0 < \alpha < \beta < 1$, then

$$L^p(\Omega,\mathbb{H}_\beta) \hookrightarrow L^p(\Omega,\mathbb{H}_\alpha) \hookrightarrow L^p(\Omega;\mathbb{H})$$

are continuously embedded and hence there exist constants $k_1 > 0$, $k(\alpha) > 0$ such that

$$\mathbf{E}\|X\|^p \le k_1 \mathbf{E}\|X\|_\alpha^p \text{ for each } X \in L^p(\Omega,\mathbb{H}_\alpha) \text{ and}$$
$$\mathbf{E}\|X\|_\alpha^p \le k(\alpha)\mathbf{E}\|X\|_\beta^p \text{ for each } X \in L^p(\Omega,\mathbb{H}_\beta).$$

$(6H)_1$ The operator $\mathscr{A}$ is sectorial and generates a hyperbolic (analytic) semigroup $(T(t))_{t\ge 0}$.

$(6H)_2$ Let $\alpha \in (0,\frac{1}{2})$. Then $\mathbb{H}_\alpha = D((-\mathscr{A})^\alpha)$, or $\mathbb{H}_\alpha = D_{\mathscr{A}}(\alpha,p)$, $1 \le p \le \infty$, or $\mathbb{H}_\alpha = D_{\mathscr{A}}(\alpha)$, or $\mathbb{H}_\alpha = [\mathbb{H}, D(\mathscr{A})]_\alpha$. We also assume that $B, C, L : L^p(\Omega,\mathbb{H}_\alpha) \to L^p(\Omega;\mathbb{H})$ are bounded linear operators and set

$$\varpi := \max\Big(\|B\|_{B(L^p(\Omega,\mathbb{H}_\alpha),\,L^p(\Omega;\mathbb{H}))}, \|C\|_{B(L^p(\Omega,\mathbb{H}_\alpha),L^p(\Omega;\mathbb{H}))}, \|L\|_{B(L^p(\Omega,\mathbb{H}_\alpha),\,L^p(\Omega;\mathbb{H}))}\Big).$$

$(6H)_3$ Let $\alpha \in (0,\frac{1}{2}-\frac{1}{p})$ if $p > 2$ and $\alpha \in (0,\frac{1}{2})$ if $p = 2$, and $\alpha < \beta < 1$. The functions $F : \mathbb{R} \times L^p(\Omega;\mathbb{H}) \to L^p(\Omega,\mathbb{H}_\beta)$, $G : \mathbb{R} \times L^p(\Omega;\mathbb{H}) \to L^p(\Omega;\mathbb{H})$ and $H : \mathbb{R} \times L^p(\Omega;\mathbb{H}) \to L^p(\Omega;\mathbb{L}_2^0)$ are p-th mean almost periodic. Moreover, the functions F, G, and H are uniformly Lipschitz with respect to the second argument in the following sense: there exist positive constants K_F, K_G, and K_H such that

$$\mathbf{E}\|F(t,X) - F(t,Y)\|_\beta^p \le K_F\,\mathbf{E}\|X - Y\|^p,$$

$$\mathbf{E}\|G(t,X) - G(t,Y)\|^p \le K_G\,\mathbf{E}\|X - Y\|^p, \text{ and}$$
$$\mathbf{E}\|H(t,X) - H(t,Y)\|_{\mathbb{L}_2^0}^p \le K_H\,\mathbf{E}\|X - Y\|^p$$

for all stochastic processes $X, Y \in L^p(\Omega;\mathbb{H})$ and $t \in \mathbb{R}$.

Theorem 6.1. *Under assumptions* $(6H)_1$, $(6H)_2$, *and* $(6H)_3$, *the evolution equation (6.1) has a unique p-th mean almost periodic mild solution whenever* $\Theta < 1$, *where*

$$\begin{aligned}\Theta := {} & k'(\alpha)\cdot K'_F\,\varpi\left(1+c\left[C(\Gamma,\alpha,\gamma,p)+\frac{2}{\delta}\right]\right)\\ & +k'_1\,C(\alpha)\,K'_G\varpi\left[C(\Gamma,\alpha,\gamma,p)+\frac{2}{\delta}\right]\\ & +2c\,C_p\cdot K'_H\,C_1(\Gamma,\xi,\delta,p)\,\varpi\Big[M_1(\alpha)^p\,C_2(\Gamma,\alpha,\xi,\delta,p)+C_3(\Gamma,\xi,\delta,p)\Big]\end{aligned}$$

for $p > 2$ *and*

$$\begin{aligned}\Theta := \varpi\Bigg[& k'(\alpha)\,K'_F\left\{1+c\left(\frac{\Gamma(1-\alpha)}{\gamma^{1-\alpha}}+\frac{1}{\delta}\right)\right\}+k'_1\cdot K'_G\left(M'(\alpha)\,\frac{\Gamma(1-\alpha)}{\gamma^{1-\alpha}}+\frac{C'(\alpha)}{\delta}\right)\\ & +c\cdot K'_H\cdot k'_1\cdot\left\{\frac{K'(\alpha,\beta)}{\sqrt{\delta}}+2K'(\alpha,\gamma,\delta,\Gamma)\right\}\Bigg]\end{aligned}$$

for $p = 2$.

To prove Theorem 6.1, we will need the following lemmas, which will be proven under our initial assumptions.

Lemma 6.1. *Under assumptions* $(6H)_1$, $(6H)_2$, *and* $(6H)_3$, *the integral operators* Γ_1 *and* Γ_2 *defined above map* $AP(\mathbb{R};L^p(\Omega,\mathbb{H}_\alpha))$ *into itself.*

Proof. The proof for the p-th mean almost periodicity of $\Gamma_2 X$ is similar to that of $\Gamma_1 X$ and hence will be omitted. Let $X \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. Since $B \in B(L^p(\Omega;\mathbb{H}_\alpha),L^p(\Omega;\mathbb{H}))$ it follows that the function $t \mapsto BX(t)$ belongs to $AP(\mathbb{R};L^p(\Omega;\mathbb{H}))$. Using Theorem 4.4 it follows that $\Psi(\cdot) = F(\cdot,BX(\cdot))$ is in $AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\beta))$ whenever $X \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. We can now show that $\Gamma_1 X \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. Indeed, since $X \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\beta))$, for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for all ξ there is $t \in [\xi,\xi+l(\varepsilon)]$ with the property

$$\mathbf{E}\Big\|\Psi X(t+\tau)-\Psi X(t)\Big\|_\beta^p < \nu\varepsilon \text{ for each } t \in \mathbb{R},$$

where

$$\nu = \frac{1}{M'(\alpha)^p\,C(\Gamma,\alpha,\gamma,p)}$$

with

$$C(\Gamma,\alpha,\gamma,p) = \left[\Gamma\left(1-\frac{p\,\alpha}{p-1}\right)\right]^{p-1}\gamma^{p(\alpha-1)}$$

and $\Gamma(\cdot)$ being the classical gamma function.
Now, the estimate in Eq. (2.33) yields

$$
\begin{aligned}
&\mathbf{E}\big\|\Gamma_1 X(t+\tau)-\Gamma_1 X(t)\big\|_\alpha^p \\
&\quad \le \mathbf{E}\left(\int_0^\infty \big\|AT(s)P[\Psi(t-s+\tau)-\Psi(t-s)]\big\|_\alpha \, ds\right)^p \\
&\quad \le M'(\alpha)\mathbf{E}\Big[\int_0^\infty s^{-\alpha}e^{-\gamma s}\big\|\Psi(t-s+\tau)-\Psi(t-s)]\big\|_\beta \, ds\Big]^p \\
&\quad \le M'(\alpha)^p\left(\int_0^\infty s^{-\frac{p\alpha}{p-1}}e^{-\gamma s}\, ds\right)^{p-1} \times \\
&\qquad \times \left(\int_0^\infty e^{-\gamma s}\mathbf{E}\big\|\Psi(t-s+\tau)-\Psi(t-s)\big\|_\beta^p \, ds\right) \\
&\quad \le M'(\alpha)^p\left(\Gamma\big(1-\frac{p\,\alpha}{p-1}\big)\gamma^{\frac{p\,\alpha}{p-1}-1}\right)^{p-1} \times \\
&\qquad \times \left(\frac{1}{\gamma}\right)\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi(t+\tau)-\Psi(t)\big\|_\beta^p \, ds \\
&\quad \le M'(\alpha)^p\, C(\Gamma,\alpha,\gamma,p)\, \sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi(t+\tau)-\Psi(t)\big\|_\beta^p \\
&\quad < \varepsilon
\end{aligned}
$$

for each $t\in\mathbb{R}$, and hence $\Gamma_1 X \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$.

Lemma 6.2. *Under assumptions* $(6H)_1$, $(6H)_2$, *and* $(6H)_3$, *the integral operators* Γ_3 *and* Γ_4 *defined above map* $AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$ *into itself.*

Proof. The proof for the p-th mean almost periodicity of $\Gamma_4 X$ is similar to that of $\Gamma_3 X$ and hence will be omitted. Note, however, that for $\Gamma_4 X$, we make use of Eq. (2.30) rather than Eq. (2.31).

Let $X\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}_\alpha))$. Since $C\in B(L^p(\Omega;\mathbb{H}_\alpha),L^p(\Omega;\mathbb{H}))$, it follows that $CX\in AP(\mathbb{R},L^p(\Omega;\mathbb{H}))$. Setting $\Phi(t)=G(t,CX(t))$ and using Theorem 4.4 it follows that $\Phi\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}))$. We can now show that $\Gamma_3 X\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}_\alpha))$. Indeed, since $\Phi\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}))$, for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that for all ξ there is $\tau\in[\xi,\xi+l(\varepsilon)]$ with

$$\mathbf{E}\big\|\Phi(t+\tau)-\Phi(t)\big\|^p < \mu\cdot\varepsilon \text{ for each } t\in\mathbb{R},$$

where $\mu = \dfrac{1}{M(\alpha)\,C(\Gamma,\alpha,\gamma,p)}$.

Now, using the expression

$$(\Gamma_3 X)(t+\tau)-(\Gamma_3 X)(t) = \int_0^\infty T(s)P[\Phi(t-s+\tau)-\Phi(t-s)]\, ds$$

and Hölder's inequality along with Eq. (2.31) yields

$$\begin{aligned}
&\mathbf{E}\big\|(\Gamma_3 X)(t+\tau)-(\Gamma_3 X)(t)\big\|_\alpha^p\\
&\qquad\le \mathbf{E}\Big[\int_0^\infty \big\|T(s)P[\Phi(t-s+\tau)-\Phi(t-s)]\big\|_\alpha\,ds\Big]^p\\
&\qquad\le M(\alpha)^p\,\mathbf{E}\Big[\int_0^\infty s^{-\alpha}\,e^{-\gamma s}\big\|\Phi(t-s+\tau)-\Phi(t-s)\big\|\,ds\Big]^p\\
&\quad\le M(\alpha)^p\,C(\Gamma,\alpha,\gamma,p)\,\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Phi(t+\tau)-\Phi(t)\big\|^p\\
&\quad<\varepsilon
\end{aligned}$$

for each $t\in\mathbb{R}$, and hence, $\Gamma_3 X\in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$.

Lemma 6.3. *Under assumptions* $(6H)_1$*,* $(6H)_2$*, and* $(6H)_3$*, the integral operators* Γ_5 *and* Γ_6 *defined above map* $AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$ *into itself.*

Proof. The proof for the p-th mean almost periodicity of $\Gamma_6 X$ is similar to that of $\Gamma_5 X$ and hence will be omitted. Note, however, that for $\Gamma_6 X$, we make use of Eq. (2.30) rather than Eq. (2.31). Let $X\in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. Since $L\in B(L^p(\Omega;\mathbb{H}_\alpha),L^p(\Omega;\mathbb{H}))$, it follows that $LX\in AP(\mathbb{R},L^p(\Omega;\mathbb{H}))$. Setting $\Lambda(t)=H(t,LX(t))$ and using Theorem 4.4 it follows that $\Lambda\in AP(\mathbb{R};L^p(\Omega;\mathbb{L}_2^0))$. We claim that $\Gamma_5 X\in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. Indeed, since $\Lambda\in AP(\mathbb{R};L^p(\Omega;\mathbb{L}_2^0))$, for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that for all ξ there is $\tau\in[\xi,\xi+l(\varepsilon)]$ with

$$\mathbf{E}\big\|\Lambda(t+\tau)-\Lambda(t)\big\|_{\mathbb{L}_2^0}^p<\zeta\cdot\varepsilon \text{ for each } t\in\mathbb{R}, \tag{6.4}$$

where

$$\zeta=\begin{cases}\frac{1}{C_pM(\alpha)^p\cdot C_2(\Gamma,\alpha,\xi,\delta,p)} & \text{if } \ p>2,\\[2ex] \dfrac{1}{M(\alpha)^2\cdot K(\Gamma,\alpha,\gamma)} & \text{if } \ p=2.\end{cases}$$

Here we split the proof into two cases: $p>2$ and $p=2$.

Now, to show the p-th mean almost periodicity of Γ_5, we break down the proof into two cases: $p>2$ and $p=2$ and use the representation (3.34) discussed in Chapter 3. We have

$$\begin{aligned}
&(\Gamma_5 X)(t+\tau)-(\Gamma_5 X)(t)\\
&\quad=\frac{\sin(\pi\xi)}{\pi}\Big[\int_{-\infty}^{t+\tau}(t+\tau-s)^{\xi-1}T(t+\tau-s)P\,\mathbb{S}_\Lambda(s)\,ds\\
&\qquad\qquad-\int_{-\infty}^{t}(t-s)^{\xi-1}T(t-s)P\,\mathbb{S}_\Lambda(s)\,ds\Big]\\
&\quad=\frac{\sin(\pi\xi)}{\pi}\int_0^\infty s^{\xi-1}T(s)P\Big[\mathbb{S}_\Lambda(t-s+\tau)-\mathbb{S}_\Lambda(t-s)\Big]\,ds
\end{aligned}$$

where

$$\mathbb{S}_\Lambda(s) = \int_{-\infty}^{s} (s-\sigma)^{-\xi} T(s-\sigma) P \Lambda(\sigma)\, d\mathbb{W}(\sigma)\,,$$

with ξ satisfying $\alpha + \frac{1}{p} < \xi < \frac{1}{2}$.

Next, let us show that the stochastic process $\mathbb{S}_\Lambda$ is p-th mean almost periodic. To this end, we proceed as in the proof of Proposition 3.26 (i) with $U(t,s) = T(t-s)$ and $P(s) = P$, and obtain

$$\begin{aligned}
&\mathbf{E}\big\|\mathbb{S}_\Lambda(t+\tau) - \mathbb{S}_\Lambda(t)\big\|^p \\
&\quad \le C_p C_1(\Gamma,\xi,\delta,p) \sup_{s\in\mathbb{R}} \mathbf{E}\big\|\Lambda(t-s+\tau) - \Lambda(t-s)\big\|^p_{\mathbb{L}_2^0} \\
&\quad \le \frac{\varepsilon}{M(\alpha)^p C_2(\Gamma,\alpha,\xi,\delta,p)}\,.
\end{aligned}$$

Hence, $\mathbb{S}_{\Psi_3}$ is p-th mean almost periodic.

We are now prepared to show the p-th mean almost periodicity of $\Gamma_5 X(\cdot)$. An application of Proposition 3.26 (ii) with $U(t,s) = T(t-s)$ and $P(s) = P$ shows that

$$\begin{aligned}
&\mathbf{E}\big\|(\Gamma_5 X)(t+\tau) - (\Gamma_5 X)(t)\big\|^p_\alpha \\
&\quad \le M(\alpha)^p C_2(\Gamma,\alpha,\xi,\delta,p) \sup_{s\in\mathbb{R}} \mathbf{E}\big\|\mathbb{S}_\Lambda(s+\tau) - \mathbb{S}_\Lambda(s)\big\|^p \\
&\quad < \varepsilon\,.
\end{aligned}$$

Hence, $\Gamma_5 X(\cdot)$ is p-th mean almost periodic.

For $p = 2$, we have

$$\begin{aligned}
&\mathbf{E}\big\|(\Gamma_5 X)(t+\tau) - (\Gamma_5 X)(t)\big\|^2_\alpha \\
&\quad = \mathbf{E}\left\|\int_0^\infty T(s)P\big[\Lambda(t-s+\tau) - \Lambda(t-s)\big]\, d\mathbb{W}(s)\right\|^2_\alpha \\
&\quad \le M(\alpha)^2 \int_0^\infty s^{-2\alpha} e^{-2\gamma s} \mathbf{E}\big\|\Lambda(t-s+\tau) - \Lambda(t-s)\big\|^2_{\mathbb{L}_2^0}\, ds \\
&\quad \le M(\alpha)^2 \left(\int_0^\infty s^{-2\alpha} e^{-2\gamma s}\, ds\right) \sup_{t\in\mathbb{R}} \mathbf{E}\big\|\Lambda(t+\tau) - \Lambda(t)\big\|^2_{\mathbb{L}_2^0} \\
&\quad \le M(\alpha)^2 K(\Gamma,\alpha,\gamma) \sup_{t\in\mathbb{R}} \mathbf{E}\big\|\Lambda(t+\tau) - \Lambda(t)\big\|^2_{\mathbb{L}_2^0} \\
&\quad < \varepsilon\,,
\end{aligned}$$

where $K(\Gamma,\alpha,\gamma) = \Gamma\Big(1-2\alpha\Big)(2\gamma)^{2\alpha-1}$, and it follows from Eq. (6.4) that $\Gamma_5 X \in AP(\mathbb{R}; L^p(\Omega;\mathbb{H}_\alpha))$.

We are now ready to prove Theorem 6.1.

Proof. Consider the nonlinear operator $\mathbb{M}$ on the space $AP(\mathbb{R}; L^p(\Omega;\mathbb{H}_\alpha))$ equipped with the α-sup norm $\|X\|_{\infty,\alpha} = \sup_{t\in\mathbb{R}} (\mathbf{E}\|X(t)\|^p_\alpha)^{1/p}$ and defined by

$$
\begin{aligned}
\mathbb{M}X(t) = -F(t,BX(t)) - \int_{-\infty}^{t} AT(t-s)PF(s,BX(s))\,ds + \int_{t}^{\infty} AT(t-s)QF(s,BX(s))\,ds \\
+ \int_{-\infty}^{t} T(t-s)P\,G(s,CX(s))\,ds - \int_{t}^{\infty} T(t-s)Q\,G(s,CX(s))\,ds \\
+ \int_{-\infty}^{t} T(t-s)P\,H(s,LX(s))\,d\mathbb{W}(s) - \int_{t}^{\infty} T(t-s)Q\,H(s,LX(s))\,d\mathbb{W}(s)
\end{aligned}
$$

for each $t \in \mathbb{R}$.

As we have previously seen, for every $X \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$, $f(\cdot,BX(\cdot)) \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\beta)) \subset AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. In view of Lemmas 6.1, 6.2, and 6.3, it follows that $\mathbb{M}$ maps $AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$ into itself. To complete the proof one has to show that $\mathbb{M}$ has a unique fixed point.

Let $X,\, Y \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. By $(6H)_1$, $(6H)_2$, and $(6H)_3$, we obtain

$$
\begin{aligned}
\mathbf{E}\big\|F(t,BX(t)) - F(t,BY(t))\big\|_\alpha^p &\le k(\alpha)K_F\mathbf{E}\big\|BX(t) - BY(t)\big\|^p \\
&\le k(\alpha)\cdot K_F\,\varpi^p\,\big\|X-Y\big\|_{\infty,\alpha}^p .
\end{aligned}
$$

Now for Γ_1 and Γ_2, we have the following evaluations:

$$
\begin{aligned}
&\mathbf{E}\big\|(\Gamma_1 X)(t) - (\Gamma_1 Y)(t)\big\|_\alpha^p \\
&\le \mathbf{E}\Big(\int_{-\infty}^{t} \big\|AT(t-s)P[F(t,BX(t)) - F(t,BY(t))]\big\|_\alpha\,ds\Big)^p \\
&\le c^p\Big(\int_{-\infty}^{t} (t-s)^{-\frac{p\alpha}{2(p-1)}}\,e^{-\gamma(t-s)}\,ds\Big)^{p-1} \times \\
&\quad \times\Big(\int_{-\infty}^{t} e^{-\gamma(t-s)}\mathbf{E}\big\|F(t,BX(t)) - F(t,BY(t))\big\|^p\,ds\Big) \\
&\le c^2\,k(\alpha)\,K_F\,\cdot C(\Gamma,\alpha,\gamma,p)\,\varpi^p\,\big\|X-Y\big\|_{\infty,\alpha}^p .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathbf{E}\big\|(\Gamma_2 X)(t) - (\Gamma_2 Y)(t)\big\|_\alpha^p \\
&\le \mathbf{E}\Big(\int_{t}^{\infty} \big\|AT(t-s)Q[F_1(s,X_s) - F_1(s,Y_s)]\big\|_\alpha\,ds\Big)^p \\
&\le c^p\Big(\int_{t}^{\infty} e^{-\delta(s-t)}\,ds\Big)^{p-1}\Big(\int_{t}^{\infty} e^{-\delta(s-t)}\mathbf{E}\big\|F(t,BX(t)) - F(t,BY(t))\big\|_\alpha^p\,ds\Big) \\
&\le c^p\,k(\alpha)\,K_F\,\cdot\,\frac{1}{\delta^p}\,\varpi^p\,\big\|X-Y\big\|_{\infty,\alpha}^p .
\end{aligned}
$$

As to Γ_3 and Γ_4, we have the following evaluations:

$$\begin{aligned}
&\mathbf{E}\big\|(\Gamma_3X)(t)-(\Gamma_3Y)(t)\big\|_\alpha^p \\
&\quad\le \mathbf{E}\left(\int_{-\infty}^t \big\|T(t-s)P[G(s,CX(s))-G(s,CY(s))]\big\|_\alpha\,ds\right)^p \\
&\quad\le C(\alpha)^p\mathbf{E}\left(\int_{-\infty}^t (t-s)^{-\alpha}e^{-\delta(t-s)}\big\|G(s,CX(s))-G(s,CY(s))\big\|\,ds\right)^p \\
&\quad\le k_1\,C(\alpha)^p\,K_G\varpi^p\,C(\Gamma,\alpha,\gamma,p)\,\big\|X-Y\big\|_{\infty,\alpha}^p\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbf{E}\big\|(\Gamma_4X)(t)-(\Gamma_4Y)(t)\big\|_\alpha^p \\
&\quad\le \mathbf{E}\Big[\int_t^\infty \big\|T(t-s)Q[G(s,CX(s))-G(s,CY(s))]\big\|_\alpha\,ds\Big]^p \\
&\quad\le C(\alpha)^p\mathbf{E}\left(\int_t^\infty e^{-\delta(s-t)}\big\|G(s,CX(s))-G(s,CY(s))\big\|_\alpha\,ds\right)^p \\
&\quad\le C(\alpha)^p\,K_G\,\varpi^p\,\frac{1}{\delta^p}\,\big\|X-Y\big\|_{\infty,\alpha}^p\,.
\end{aligned}$$

Finally for Γ_5 and Γ_6, we have the following evaluations:

$$\begin{aligned}
&(\Gamma_5X)(t)-(\Gamma_5Y)(t) \\
&=\frac{\sin(\pi\xi)}{\pi}\Big[\int_{-\infty}^t (t-s)^{\xi-1}T(t-s)P\big[\mathbb{S}_{\Lambda(X)}(s)-\mathbb{S}_{\Lambda(Y)}(s)\big]\,ds\Big] \\
&=\frac{\sin(\pi\xi)}{\pi}\Big[\int_0^\infty (t-s)^{\xi-1}T(t-s)P\big[\mathbb{S}_{\Lambda(X)}(s)-\mathbb{S}_{\Lambda(Y)}(s)\big]\,ds\Big]
\end{aligned}$$

where

$$\mathbb{S}_{\Lambda(X)}(s)=\int_{-\infty}^s (s-\sigma)^{-\xi}T(s-\sigma)P\Lambda(X)(\sigma)\,d\mathbb{W}(\sigma)\,,$$

with $\Lambda(X)(t)=H(t,L(t))$ and ξ satisfies $\alpha+\frac{1}{p}<\xi<\frac{1}{2}$.

To evaluate $\mathbf{E}\big\|(\Gamma_5X)(t)-(\Gamma_5Y)(t)\big\|_\alpha^p$, we break down the calculations into two cases: $p>2$ and $p=2$. Let us first evaluate $\mathbf{E}\big\|\mathbb{S}_{\Lambda(X)}(s)-\mathbb{S}_{\Lambda(Y)}(s)\big\|_\alpha^p$.

A direct application of Proposition 3.26 (i) with $U(t,s)=T(t-s)$ and $P(s)=P$ yields

$$\begin{aligned}
&\mathbf{E}\big\|\mathbb{S}_\Lambda(t+\tau)-\mathbb{S}_\Lambda(t)\big\|^p \\
&\quad\le C_pC_1(\Gamma,\xi,\delta,p)\sup_{s\in\mathbb{R}}\mathbf{E}\big\|\Lambda(t-s+\tau)-\Lambda(t-s)\big\|_{\mathbb{L}_2^0}^p \\
&\quad\le C_p\,C_1(\Gamma,\xi,\delta,p)\,K_H\,\varpi^p\,\big\|X-Y\big\|_{\alpha,\infty}^p.
\end{aligned}$$

We are now prepared to evaluate $\mathbf{E}\big\|(\Gamma_5 X)(t)-(\Gamma_5 Y)(t)\big\|_\alpha^p$. As before, an application of Proposition 3.26 (ii) with $U(t,s)=T(t-s)$ and $P(s)=P$ shows that

$$\begin{aligned}&\mathbf{E}\big\|(\Gamma_5 X)(t)-(\Gamma_5 Y)(t)\big\|_\alpha^p\\&\quad\le M(\alpha)^p\,C_2(\Gamma,\alpha,\xi,\delta,p)\,C_p\,C_1(\Gamma,\xi,\delta,p)\,K_H\,\varpi^p\,\big\|X-Y\big\|_{\alpha,\infty}^p.\end{aligned}$$

For $p=2$, we have

$$\begin{aligned}&\mathbf{E}\big\|(\Gamma_5 X)(t)-(\Gamma_5 Y)(t)\big\|_\alpha^2\\&=\mathbf{E}\left\|\int_{-\infty}^t T(t-s)P\big[H(s,LX(s))-H(s,LY(s))\big]\,d\mathbb{W}(s)\right\|_\alpha^2\\&M(\alpha)^2\int_{-\infty}^t (t-s)^{-2\alpha}e^{-2\gamma(t-s)}\mathbf{E}\big\|H(s,LX(s))-H(s,LY(s))\big\|_{\mathbb{L}_2^0}^2\,ds\\&\le M(\alpha)^2\left(\int_{-\infty}^t (t-s)^{-2\alpha}e^{-2\gamma(t-s)}\,ds\right)\sup_{t\in\mathbb{R}}\mathbf{E}\big\|H(t,LX(t))-H(t,LY(t))\big\|_{\mathbb{L}_2^0}^2\\&\le K_H\,M(\alpha)^2\,K(\Gamma,\alpha,\gamma)\,\varpi^2\,\big\|X-Y\big\|_{\alpha,\infty}^2.\end{aligned}$$

Similarly, analogous computations used in the proof of Proposition 3.26 with $U(t,s)=T(t-s)$ and $Q(s)=Q$ show that

$$\begin{aligned}&\mathbf{E}\big\|(\Gamma_6 X)(t)-(\Gamma_6 Y)(t)\big\|_\alpha^p\\&\quad\le M(\alpha)^p\,C_3(\Gamma,\xi,\delta,p)\,C_p\,C_1(\Gamma,\xi,\delta,p)\,K_H\,\varpi^p\,\big\|X-Y\big\|_{\alpha,\infty}^p,\end{aligned}$$

and this is valid when $p>2$.

For $p=2$, we have

$$\begin{aligned}&\mathbf{E}\big\|(\Gamma_6 X)(t)-(\Gamma_6 Y)(t)\big\|_\alpha^2\\&\le\mathbf{E}\left\|\int_{-\infty}^t T(t-s)Q\big[H(s,LX(s))-H(s,LY(s))\big]\,d\mathbb{W}(s)\right\|_\alpha^2\\&\le C(\alpha)^2\mathbf{E}\int_{-\infty}^t e^{2\delta(t-s)}\mathbf{E}\big\|H(s,LX(s))-H(s,LY(s))\big\|^2\,ds\\&\le C(\alpha)^2\left(\int_{-\infty}^t e^{2\delta(t-s)}\,ds\right)\sup_{s\in\mathbb{R}}\mathbf{E}\big\|H(s,LX(s))-H(s,LY(s))\big\|^2\\&\le C(\alpha)^2\,K_H\,\frac{1}{2\delta}\,\varpi^2\,\big\|X-Y\big\|_{\alpha,\infty}^2.\end{aligned}$$

Consequently,

$$\big\|\mathbb{M}X-\mathbb{M}Y\big\|_{\infty,\alpha}\le\Theta\cdot\big\|X-Y\big\|_{\infty,\alpha}.$$

Clearly, if $\Theta<1$, then Eq. (6.1) has a unique fixed-point by the Banach fixed-point theorem, which is obviously the only p-th mean almost periodic solution to it.

6.1.2 Example

Let $\Gamma \subset \mathbb{R}^N$ ($N \geq 1$) be an open bounded subset with C^2 boundary $\partial\Gamma$. To illustrate our abstract results, we study the existence of square-mean almost periodic solutions to the stochastic heat equation given by

$$\begin{cases} \partial_t \Big[\Phi + F(t, D_x \Phi)\Big] = \partial_t \Big[\Delta \Phi + G(t, D_x \Phi)\Big] + H(t,\Phi)\partial_t \mathbb{W}(t), \text{ in } \Gamma \\ \Phi = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Gamma \end{cases} \tag{6.5}$$

where the unknown Φ is a function of $\omega \in \Omega$, $t \in \mathbb{R}$, and $x \in \Gamma$, the symbols D_x and Δ stand respectively for the differential operators defined by

$$D_x = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \ \text{ and } \Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2},$$

and the coefficients $F, G : \mathbb{R} \times L^2(\Omega, H_0^\alpha(\Gamma) \cap H^{2\alpha}(\Gamma)) \to L^2(\Omega, L^2(\Gamma))$ and $H : \mathbb{R} \times L^2(\Omega, H_0^\alpha(\Gamma) \cap H^{2\alpha}(\Gamma)) \to L^2(\Omega, L^2(\Gamma))$ are square-mean almost periodic, and $\mathbb{W}$ is one-dimensional Brownian motion.

Define the linear operator appearing in Eq. (6.5) as follows:

$$AX = \Delta X \quad \text{for all } \ u \in D(A) = L^2(\Omega; H_0^1(\Gamma) \cap H^2(\Gamma)).$$

Using the fact that, the operator $\mathscr{A}$ defined in $L^2(\Gamma)$ by

$$\mathscr{A}u = \Delta u \quad \text{for all } \ u \in D(\mathscr{A}) = H_0^1(\Gamma) \cap H^2(\Gamma)$$

is sectorial and whose corresponding analytic semigroup is hyperbolic, one can easily sees that the operator A defined above is sectorial and hence is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$.

For each $\mu \in (0,1)$, we take $\mathbb{H}_\mu = D((-\Delta)^\mu) = L^2(\Omega, H_0^\mu(\Gamma) \cap H^{2\mu}(\Gamma))$ equipped with its μ-norm $\|\cdot\|_\mu$. Moreover, since $\alpha \in (0, \frac{1}{2})$, we suppose that $\frac{1}{2} < \beta < 1$. Letting $L = I$, and $BX = CX = D_x X$ for all $X \in L^2(\Omega, \mathbb{H}_\alpha) = L^2(\Omega, D((-\Delta)^\alpha))$ $= L^2(\Omega, H_0^\alpha(\Gamma) \cap H^{2\alpha}(\Gamma))$, one easily see that both B and C are bounded from $L^2(\Omega, H_0^\alpha(\Gamma) \cap H^{2\alpha}(\Gamma))$ in $L^2(\Omega, L^2(\Gamma))$ with $\varpi = 1$.

We require the following assumption:

$(6H)_4$ Let $\frac{1}{2} < \beta < 1$, and $F : \mathbb{R} \times L^2(\Omega, H_0^\alpha(\Gamma) \cap H^{2\alpha}(\Gamma)) \to L^2(\Omega, H_0^\beta(\Gamma) \cap H^{2\beta}(\Gamma))$ be square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in L^2(\Omega, H_0^\alpha(\Gamma) \cap H^{2\alpha}(\Gamma))$, $G : \mathbb{R} \times L^2(\Omega, H_0^\alpha(\Gamma) \cap H^{2\alpha}(\Gamma)) \to L^2(\Omega, L^2(\Gamma))$ be square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in L^2(\Omega, H_0^\alpha(\Gamma) \cap H^{2\alpha}(\Gamma))$. Moreover, the functions F, G are uniformly Lipschitz with respect to the second argument in the following sense: there exists $K' > 0$ such that

$$\mathbf{E}\left\|F(t,\Phi_1)-F(t,\Phi_2)\right\|_\beta^2 \leq K'\mathbf{E}\left\|\Phi_1-\Phi_2\right\|_{L^2(\Gamma)}^2,$$

$$\mathbf{E}\left\|G(t,\Phi_1)-G(t,\Phi_2)\right\|_{L^2(\Gamma)}^2 \leq K'\mathbf{E}\left\|\Phi_1-\Phi_2\right\|_{L^2(\Gamma)}^2,$$

and

$$\mathbf{E}\left\|H(t,\psi_1)-H(t,\psi_2)\right\|_{L^2(\Gamma)}^2 \leq K'\mathbf{E}\left\|\psi_1-\psi_2\right\|_{L^2(\Gamma)}^2$$

for all $\Phi_1,\Phi_2,\psi_1,\psi_2 \in L^2(\Omega;L^2(\Gamma))$ and $t\in\mathbb{R}$.

We have

Theorem 6.2. *Under the previous assumptions including $(6H)_4$, then the N-dimensional stochastic heat equation (6.5) has a unique square-mean almost periodic solution $\Phi \in L^2(\Omega, H_0^1(\Gamma)\cap H^2(\Gamma))$ whenever K' is small enough.*

6.2 The Nonautonomous Case

6.2.1 Introduction

In this section, we consider a more general setting, that is, we make extensive use of intermediate space techniques to study the existence of p-th mean almost periodic solutions to the class of nonautonomous stochastic differential equations given by

$$\begin{aligned} d\Big(X(\omega,t)+f_1(t,\mathscr{B}X(\omega,t))\Big) &= \Big[\mathscr{A}(t)X(\omega,t)+f_2(t,\mathscr{C}X(\omega,t))\Big]dt \\ &\quad + f_3(t,\mathscr{L}X(\omega,t))d\mathbb{W}(\omega,t) \end{aligned} \tag{6.6}$$

for all $t\in\mathbb{R}$ and $\omega\in\Omega$, where $\mathscr{A}(t)$ for $t\in\mathbb{R}$ is a family of closed linear operators on $\mathscr{D}=D(\mathscr{A}(t))$, which is independent of t, satisfying the well-known Acquistapace and Terreni conditions (2.38)-(2.39), $f_1:\mathbb{R}\times\mathbb{H}\to\mathbb{H}_\beta^t$ $(0<\alpha<\frac{1}{p}<\beta<1)$, $f_2:\mathbb{R}\times\mathbb{H}\to\mathbb{H}$, and $f_3:\mathbb{R}\times\mathbb{H}\to\mathbb{L}_2^0$ are p-th mean almost periodic in $t\in\mathbb{R}$ uniformly in the second variable. It is well known that in that case, there exists an evolution family $\mathscr{U}=\{U(t,s)\}_{t\geq s}$ associated with the family of operators $\mathscr{A}(t)$. Assuming that the evolution family $\mathscr{U}=\{U(t,s)\}_{t\geq s}$ is exponentially dichotomic (hyperbolic) and under some additional assumptions it will be shown that Eq. (6.6) has a unique p-th mean almost periodic solution.

To analyze Eq. (6.6), our strategy consists in studying the existence of p-th mean almost periodic solutions to the corresponding class of stochastic differential equations of the form

$$\begin{aligned} d\Big(X(t)+F_1(t,BX(t))\Big) &= \Big[A(t)X(t)+F_2(t,CX(t))\Big]dt \\ &\quad +F_3(t,LX(t))d\mathbb{W}(t) \end{aligned} \tag{6.7}$$

for all $t \in \mathbb{R}$, where $A(t) : D(A(t)) \subset L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{H})$ is a sectorial linear operator whose corresponding analytic semigroup is hyperbolic, that is, $\sigma(A(t)) \cap i\mathbb{R} = \emptyset$, B, C, and L are (possibly unbounded linear operators on $L^p(\Omega,\mathbb{H})$) and $F_1 : \mathbb{R} \times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{H}^t_\beta)$ ($0 < \alpha < \frac{1}{2} - \frac{1}{p}$ and $\alpha < \beta < 1$), $F_2 : \mathbb{R} \times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{H})$, $F_3 : \mathbb{R} \times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{L}^0_2)$ are jointly continuous functions satisfying some additional assumptions, and $\mathbb{W}(t)$ is a $\mathscr{Q}$-Wiener process with values in $\mathbb{K}$.

6.2.2 Existence of *p*-th Mean Almost Periodic Solutions

In the present work we study operators $\mathscr{A}(t)$ on $\mathbb{H}$ subject to Acquistapace–Terreni conditions (2.38)–(2.39). In addition, we also need the following assumptions:

$(6H)_5$ The evolution family $\mathscr{U}$ generated by $\mathscr{A}(\cdot)$ has an exponential dichotomy with constants N, $\delta > 0$ and dichotomy projections $P(t)$ for $t \in \mathbb{R}$. Moreover, $0 \in \rho(\mathscr{A}(t))$ for each $t \in \mathbb{R}$ and the following holds:

$$\sup_{t,s\in\mathbb{R}} \left\|\mathscr{A}(s)\mathscr{A}^{-1}(t)\right\|_{B(\mathbb{H},\mathbb{H}_\alpha)} < c_0 \,. \tag{6.8}$$

$(6H)_6$ There exist $0 \le \alpha < \beta < 1$ and $t_0 \in \mathbb{R}$ such that

$$\mathbb{H}^t_\alpha = \mathbb{H}^{t_0}_\alpha \ \text{ and } \ \mathbb{H}^t_\beta = \mathbb{H}^{t_0}_\beta$$

for all $t \in \mathbb{R}$, with uniform equivalent norms.

Definition 6.2. Let $\alpha \in (0,1)$. A continuous random function, $X : \mathbb{R} \to L^p(\Omega;\mathbb{H}_\alpha)$ is said to be a mild solution of Eq. (6.7) provided that the function $s \to \mathbf{E}\left\|A(s)U(t,s)P(s)F_1(s,BX(s))\right\|^p$ is integrable on $(-\infty,t)$, the function $s \to \mathbf{E}\left\|A(s)U(t,s)Q(s)F_1(s,BX(s))\right\|^p$ is integrable on (t,∞) for each $t \in \mathbb{R}$, and

$$\begin{aligned} X(t) = -F_1(t,BX(t)) &- \int_{-\infty}^t A(s)U(t,s)P(s)\,F_1(s,BX(s))\,ds \\ &+ \int_t^\infty A(s)U(t,s)Q(s)\,F_1(s,BX(s))\,ds \\ &+ \int_{-\infty}^t U(t,s)P(s)\,F_2(s,CX(s))\,ds - \int_t^\infty U(t,s)Q(s)\,F_2(s,CX(s))\,ds \\ &+ \int_{-\infty}^t U(t,s)P(s)\,F_3(s,LX(s))\,d\mathbb{W}(s) - \int_t^\infty U(t,s)Q(s)\,F_3(s,LX(s))\,d\mathbb{W}(s) \end{aligned}$$

for each $t \in \mathbb{R}$.

Throughout the rest of the paper we denote by Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, and Γ_6 the nonlinear integral operators defined by

$$
\begin{aligned}
(\Gamma_1 X)(t) &:= \int_{-\infty}^{t} A(s)U(t,s)P(s)\,\Psi_1(s)\,ds,\\
(\Gamma_2 X)(t) &:= \int_{t}^{\infty} A(s)U(t,s)Q(s)\,\Psi_1(s)\,ds,\\
(\Gamma_3 X)(t) &:= \int_{-\infty}^{t} U(t,s)P(s)\,\Psi_2(s)\,ds,\\
(\Gamma_4 X)(t) &:= \int_{t}^{\infty} U(t,s)Q(s)\,\Psi_2(s)\,ds,\\
(\Gamma_5 X)(t) &:= \int_{-\infty}^{t} U(t,s)P(s)\,\Psi_3(s)\,d\mathbb{W}(s), \ \text{ and}\\
(\Gamma_6 X)(t) &:= \int_{t}^{\infty} U(t,s)Q(s)\,\Psi_3(s)\,d\mathbb{W}(s),
\end{aligned}
$$

where $\Psi_1(t) = F_1(t,BX(t))$, $\Psi_2(t) = F_2(t,CX(t))$, and $\Psi_3(t,LX(t))$.

To discuss the existence of p-th mean almost periodic solution to Eq. (6.6) we need to set some assumptions on A, B, C, L, $F_i (i = 1,2,3)$.

$(6H)_7$ $R(\zeta,A(\cdot)) \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. Moreover, there exists a function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma \in L^1[0,\infty)$ such that for every $\varepsilon > 0$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a τ with the property

$$\big\|A(t+\tau)\Gamma(t+\tau,s+\tau) - A(t)\Gamma(t,s)\big\| \leq \varepsilon\gamma\big(\,|t-s|\,\big)$$

for all $s,t \in \mathbb{R}$.

$(6H)_8$ Let $\alpha \in (0, \frac{1}{2} - \frac{1}{p})$ if $p > 2$ and $\alpha \in (0, \frac{1}{2})$ if $p = 2$, and $\alpha < \beta < 1$ with $2\beta > \alpha + 1$. The functions $F_1 : \mathbb{R} \times L^p(\Omega;\mathbb{H}) \to L^p(\Omega,\mathbb{H}_\beta)$, $F_2 : \mathbb{R} \times L^p(\Omega;\mathbb{H}) \to L^p(\Omega;\mathbb{H})$, and $F_3 : \mathbb{R} \times L^p(\Omega;\mathbb{H}) \to L^p(\Omega;\mathbb{L}_2^0)$ are p-th mean almost periodic. Moreover, the functions $F_i (i = 1,2,3)$ are uniformly Lipschitz with respect to the second argument in the following sense: there exist positive constants $K_i (i = 1,2,3)$ such that

$$
\begin{aligned}
\mathbf{E}\big\|F_1(t,X) - F_2(t,Y)\big\|_\beta^p &\leq K_1\,\mathbf{E}\big\|X-Y\big\|^p,\\
\mathbf{E}\big\|F_2(t,X) - F_2(t,Y)\big\|^p &\leq K_2\,\mathbf{E}\big\|X-Y\big\|^p,\\
\mathbf{E}\big\|F_3(t,X) - F_3(t,Y)\big\|_{\mathbb{L}_2^0}^p &\leq K_3\,\mathbf{E}\big\|X-Y\big\|^p,
\end{aligned}
$$

for all stochastic processes $X,Y \in L^p(\Omega;\mathbb{H})$ and $t \in \mathbb{R}$.

Theorem 6.3. *Under assumptions $(5H)_1$, $(6H)_5$, $(6H)_6$, $(6H)_7$, and $(6H)_8$, the evolution equation (6.6) has a unique p-th mean almost periodic mild solution whenever $\Theta < 1$, where*

$$\Theta := \Bigg[k_1' K_1' \Big(k'(\alpha) + \frac{m(\alpha,\beta)}{\delta} + n(\alpha) C_2(\Gamma,\alpha,\delta,\xi,p)\Big)$$
$$+k_2' K_2' \Big(C_2(\Gamma,\alpha,\delta,\xi,p) + \frac{m(\alpha,\beta)}{\delta}\Big)$$
$$+C_p' k_3' K_3' \Big(M(\alpha) C_2(\Gamma,\alpha,\xi,\delta,p) + \frac{m(\alpha,\beta)}{\delta}\Big) C_1(\Gamma,\xi,\delta,p)\Bigg]\varpi$$

for $p > 2$ and

$$\Theta := \Bigg[k'(\alpha)\cdot K_1' + k_1' \cdot K_1' \Big(K(\alpha,\delta,\Gamma) + \frac{m(\alpha,\beta)}{\delta}\Big)$$
$$+k_2' \cdot K_2' \Big(K(\alpha,\delta,\Gamma) + \frac{m'(\alpha,\beta)}{\delta}\Big)$$
$$+k_3' \cdot K_3' \Big(K(\alpha,\delta,\Gamma) + \frac{K(\alpha,\beta)}{\sqrt{\delta}}\Big)\Bigg]\varpi$$

for $p = 2$.

To prove Theorem 6.3, we will need the following lemmas, which will be proven under our initial assumptions.

Lemma 6.4. *Under assumptions $(5H)_1$, $(6H)_5$, $(6H)_6$, $(6H)_7$, and $(6H)_8$, the operators Γ_1 and Γ_2 defined above map $AP(\mathbb{R};L^p(\Omega,\mathbb{H}_\alpha))$ into itself.*

Proof. The proof for the p-th mean almost periodicity of $\Gamma_2 X$ is similar to that of $\Gamma_1 X$ and hence will be omitted. Let $X \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. Since $B \in B(L^p(\Omega;\mathbb{H}_\alpha), L^p(\Omega;\mathbb{H}))$ it follows that the function $t \to BX(t)$ belongs to $AP(\mathbb{R};L^p(\Omega;\mathbb{H}))$. Using Theorem 4.4 it follows that $\Psi_1(\cdot) = F_1(\cdot, BX(\cdot))$ is in $AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\beta))$ whenever $X \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. We can now show that $\Gamma_1 X \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. Indeed, since $X \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\beta))$, for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for all ξ there is $t \in [\xi, \xi + l(\varepsilon)]$ with the property

$$\mathbf{E}\big\|\Psi_1 X(t+\tau) - \Psi_1 X(t)\big\|_\beta^p < \eta \text{ for each } t \in \mathbb{R}.$$

Now, we have

$$\begin{aligned}
&\mathbf{E}\big\|\Gamma_1 X(t+\tau) - \Gamma_1 X(t)\big\|_\alpha^p \\
&\le 2^{p-1}\mathbf{E}\Big\|\int_{-\infty}^t A(s+\tau)U(t+\tau,s+\tau)P(s+\tau)[\Psi_1(t-s+\tau) - \Psi_1(t-s)]\,ds\Big\|_\alpha^p \\
&\quad +2^{p-1}\mathbf{E}\Big\|\int_{-\infty}^t \big[A(s+\tau)U(t+\tau,s+\tau)P(s+\tau) - A(s)U(t,s)P(s)\big]\Psi_1(s)\,ds\Big\|_\alpha^p \\
&\le 2^{p-1}L_1 + 2^{p-1}L_2\,.
\end{aligned}$$

Using Eq. (2.52) it follows that

$$
\begin{aligned}
L_1 &\leq \mathbf{E}\Big\{\int_{-\infty}^{t} \big\|A(s+\tau)U(t+\tau,s+\tau)P(s+\tau)[\Psi_1(t-s+\tau)-\Psi_1(t-s)]\big\|_{\alpha}\Big\}^{p}\\
&\leq n(\alpha,\beta)^p\,\mathbf{E}\Big\{\int_{-\infty}^{t} (t-s)^{-\alpha}\,e^{-\frac{\delta}{2}(t-s)}\big\|\Psi_1(t-s+\tau)-\Psi_1(t-s)\big\|\Big\}^{p}\\
&\leq n(\alpha,\beta)^p\,C(\Gamma,\alpha,\delta,p)\,\sup_t \mathbf{E}\big\|\Psi_1(t+\tau)-\Psi_1(t)\big\|_{\beta}^{p}\\
&\leq n(\alpha,\beta)^p\,C(\Gamma,\alpha,\delta,p)\eta\,.
\end{aligned}
$$

Similarly, using assumption $(6H)_7$, it follows that

$$
\begin{aligned}
L_2 &\leq \mathbf{E}\Big\{\int_{-\infty}^{t} \big\|A(s+\tau)U(t+\tau,s+\tau)P(s+\tau)-A(s)U(t,s)P(s)\big\|\,\big\|\Psi_1(s)\big\|_{\alpha}\,ds\Big\}^{p}\\
&\leq \varepsilon^p\Big\{\int_{-\infty}^{t}\gamma(t-s)\,ds\Big\}^{p-1}\Big\{\int_{-\infty}^{t}\gamma(t-s)\big\|\Psi_1(s)\big\|_{\alpha}^{p}\,ds\Big\}\\
&\leq \varepsilon^p\Big(\int_{-\infty}^{t}\gamma(t-s)\,ds\Big)^{p}\sup_t \mathbf{E}\big\|\Psi_1(s)\big\|_{\alpha}^{p}\\
&= \varepsilon^p k(\alpha)\big\|\gamma\big\|_{L^1}^{p}\sup_t \mathbf{E}\big\|\Psi_1(s)\big\|_{\beta}^{p}\,.
\end{aligned}
$$

Therefore,

$$
\mathbf{E}\big\|\Gamma_1 X(t+\tau)-\Gamma_1 X(t)\big\|_{\alpha}^{p} \leq \Big(1+k(\alpha)\big\|\gamma\big\|_{L^1}^{p}\,\big\|\Psi_1\big\|_{\infty,\beta}^{p}\Big)\varepsilon,
$$

for each $t\in\mathbb{R}$, and hence $\Gamma_1 X\in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$.

Lemma 6.5. *Under assumptions $(5H)_1$, $(6H)_5$, $(6H)_6$, $(6H)_7$, and $(6H)_8$, the integral operators Γ_3 and Γ_4 defined above map $AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$ into itself.*

Proof. The proof for the p-th mean almost periodicity of $\Gamma_4 X$ is similar to that of $\Gamma_3 X$ and hence will be omitted. Note, however, that for $\Gamma_4 X$, we make use of Eq. (2.46) rather than Eq. (2.45).

Let $X\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}_\alpha))$. Since $C\in B(L^p(\Omega;\mathbb{H}_\alpha),L^p(\Omega;\mathbb{H}))$, it follows that $CX\in AP(\mathbb{R},L^p(\Omega;\mathbb{H})))$. Setting $\Psi_2(t)=F_2(t,CX(t))$ and using Theorem 4.4 it follows that $\Psi_2\in AP(\mathbb{R};L^p(\Omega,\mathbb{H})))$. We can now show that $\Gamma_3 X\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}_\alpha))$. Indeed, since $\Psi_2\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}))$, for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that for all ξ there is $\tau\in[\xi,\xi+l(\varepsilon)]$ with

$$
\mathbf{E}\big\|\Psi_2(t+\tau)-\Psi_2(t)\big\|^{p}<\eta \text{ for each } t\in\mathbb{R}\,.
$$

We have

$$
\begin{aligned}
&\mathbf{E}\left\|(\Gamma_3 X)(t+\tau)-(\Gamma_3 X)(t)\right\|_\alpha^p\\
&=\mathbf{E}\left\|\int_{-\infty}^t U(t+\tau,s+\tau)P(s+\tau)\Psi_2(s+\tau)\,ds-\int_{-\infty}^t U(t,s)P(s)\Psi_2(s)\,ds\right\|_\alpha^p\\
&\le 3^{p-1}\mathbf{E}\left\|\int_0^\infty U(t+\tau,t-s+\tau)P(t-s+\tau)\Big[\Psi_2(t-s+\tau)-\Psi_2(t-s)\Big]\,ds\right\|_\alpha^p\\
&\quad+3^{p-1}\mathbf{E}\left\|\int_\varepsilon^\infty\big[U(t+\tau,t-s+\tau)P(t-s+\tau)-U(t,t-s)P(t-s)\big]\Psi_2(t-s)\,ds\right\|_\alpha^p\\
&\quad+3^{p-1}\mathbf{E}\left\|\int_0^\varepsilon\big[U(t+\tau,t-s+\tau)P(t-s+\tau)-U(t,t-s)P(t-s)\big]\Psi_2(t-s)\,ds\right\|_\alpha^p\\
&\le 3^{p-1}L_1'+3^{p-1}L_2'+3^{p-1}L_3'.
\end{aligned}
$$

Using Eq. (2.45), it follows that

$$
\begin{aligned}
L_1' &\le \mathbf{E}\Big\{\int_0^\infty \big\|U(t+\tau,t-s+\tau)P(t-s+\tau)[\Psi_2(t-s+\tau)-\Psi_2(t-s)]\big\|_\alpha\,ds\Big\}^p\\
&\le c(\alpha)^p\mathbf{E}\Big\{\int_0^\infty s^{-\alpha}e^{-\frac{\delta}{2}s}\big\|\Psi_2(t-s+\tau)-\Psi_2(t-s)\big\|\,ds\Big\}^p\\
&\le c(\alpha)^p\,C(\Gamma,\alpha,\delta,p)\sup_t\mathbf{E}\big\|\Psi_2(t+\tau)-\Psi_2(t)\big\|^p\\
&\le c(\alpha)^p\,C(\Gamma,\alpha,\delta,p)\eta\,.
\end{aligned}
$$

For L_2', we use Lemma 5.1 to obtain

$$
\begin{aligned}
L_2' &\le \mathbf{E}\Big\{\int_\varepsilon^\infty \big\|\big[U(t+\tau,t-s+\tau)P(t-s+\tau)-U(t,t-s)P(t-s)\big]\Psi_2(t-s)\big\|_\alpha\,ds\Big\}^p\\
&\le \frac{2^p}{\delta^p}\,\varepsilon^p\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi_2(t)\big\|^p\,.
\end{aligned}
$$

The evaluation of the last term is straightforward. We obtain

$$
\begin{aligned}
L_3' &\le \mathbf{E}\Big\{\int_0^\varepsilon \big\|[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_2(t-s)\big\|_\alpha\,ds\Big\}^p\\
&\le 2^p\,M^p\,\varepsilon^p\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi_2(t)\big\|^p\,.
\end{aligned}
$$

Combining these evaluations, we conclude that $\Gamma_3 X\in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$.

Lemma 6.6. *Under assumptions* $(5H)_1$, $(6H)_5$, $(6H)_6$, $(6H)_7$ *and* $(6H)_8$*, the integral operators* Γ_5 *and* Γ_6 *defined above map* $AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$ *into itself.*

Proof. Let $X\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}_\alpha))$. Since $L\in B(L^p(\Omega;\mathbb{H}_\alpha),L^p(\Omega;\mathbb{H}))$, it follows that $LX\in AP(\mathbb{R},L^p(\Omega;\mathbb{H}))$. Setting $\Psi_3(t)=F_3(t,LX(t))$ and using Theorem 4.4 it follows that $\Psi_3\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}))$. We can now show that $\Gamma_5 X\in AP(\mathbb{R};L^p(\Omega,\mathbb{H}_\alpha))$. Indeed, since $\Psi_3\in AP(\mathbb{R};L^p(\Omega,\mathbb{L}_2^0))$, for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that for all ξ there is $\tau\in[\xi,\xi+l(\varepsilon)]$ with

$$\mathbf{E}\big\|\Psi_3(t+\tau)-\Psi_3(t)\big\|_{\mathbb{L}_2^0}^p < \eta \text{ for each } t \in \mathbb{R}\,.$$

To show the p-th mean almost periodicity of $(\Gamma_5 X)$, we proceed in two cases: $p > 2$ and $p = 2$.

For $p > 2$, using the representation in (3.34) we can write

$$\begin{aligned}
&(\Gamma_5 X)(t+\tau)-(\Gamma_5 X)(t)\\
&= \frac{\sin(\pi\xi)}{\pi}\Bigg[\int_{-\infty}^{t+\tau}(t+\tau-s)^{\xi-1}U(t+\tau,s)P(s)\mathbb{S}_{\Psi_3}(s)\,ds\\
&\qquad -\int_{-\infty}^{t}(t-s)^{\xi-1}U(t,s)P(s)\mathbb{S}_{\Psi_3}(s)\,ds\Bigg]\\
&= \frac{\sin(\pi\xi)}{\pi}\Bigg\{\int_0^\infty s^{\xi-1}U(t+\tau,t-s+\tau)P(t-s+\tau)\Big[\mathbb{S}_{\Psi_3}(t-s+\tau)-\mathbb{S}_{\Psi_3}(t-s)\Big]\,ds\\
&+\int_0^\infty s^{\xi-1}\Big[U(t+\tau,t-s+\tau)P(t-s+\tau)-U(t,t-s)P(t-s)\Big]\mathbb{S}_{\Psi_3}(t-s)\,ds\Bigg\}
\end{aligned}$$

where

$$\mathbb{S}_{\Psi_3}(s)=\int_{-\infty}^{s}(s-\sigma)^{-\xi}U(s,\sigma)P(\sigma)\Psi_3(\sigma)\,d\mathbb{W}(\sigma)\,,$$

with ξ satisfying $\alpha+\frac{1}{p}<\xi<\frac{1}{2}$.

Let us first show that the stochastic process $\mathbb{S}_{\Psi_3}$ is p-th mean almost periodic.

An application of Proposition 3.26 (i) shows that

$$\begin{aligned}
&\mathbf{E}\big\|\mathbb{S}_{\Psi_3}(t+\tau)-\mathbb{S}_{\Psi_3}(t)\big\|^p\\
&\quad\le C_pC_1(\Gamma,\xi,\delta,p)\sup_{s\in\mathbb{R}}\mathbf{E}\big\|\Psi_3(t-s+\tau)-\Psi_3(t-s)\big\|_{\mathbb{L}_2^0}^p\\
&\quad\le C_pC_1(\Gamma,\xi,\delta,p)\eta\,.
\end{aligned}$$

Hence, $\mathbb{S}_{\Psi_3}$ is p-th mean almost periodic.

We are now ready to show the p-th mean almost periodicity of $\Gamma_5 X(\cdot)$.

$$
\begin{aligned}
&\mathbf{E}\left\|(\Gamma_5 X)(t+\tau)-(\Gamma_5 X)(t)\right\|_\alpha^p\\
&\le 3^{p-1}\left|\frac{\sin(\pi\xi)}{\pi}\right|^p \times\\
&\times \mathbf{E}\left\|\int_0^\infty s^{\xi-1}U(t+\tau,t-s+\tau)P(t-s+\tau)\left[\mathbb{S}_{\Psi_3}(t-s+\tau)-\mathbb{S}_{\Psi_3}(t-s)\right]ds\right\|_\alpha^p\\
&+3^{p-1}\mathbf{E}\left\|\int_\varepsilon^\infty s^{\xi-1}\left[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\right]\mathbb{S}_{\Psi_3}(t-s)\,ds\right\|_\alpha^p\\
&+3^{p-1}\mathbf{E}\left\|\int_0^\varepsilon s^{\xi-1}\left[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\right]\mathbb{S}_{\Psi_3}(t-s)\,ds\right\|_\alpha^p\\
&\le 3^{p-1}L_1''+3^{p-1}L_2''+3^{p-1}L_3''.
\end{aligned}
$$

Using Eq. (2.45) and subsequently applying Proposition 3.26 yields

$$
\begin{aligned}
L_1'' &\le \mathbf{E}\left\{\int_0^\infty s^{\xi-1}\left\|U(t+\tau,t-s+\tau)P(t-s+\tau)[\mathbb{S}_{\Psi_3}(t-s+\tau)-\mathbb{S}_{\Psi_3}(t-s)]\right\|_\alpha ds\right\}^p\\
&\le c(\alpha)^2\mathbf{E}\left\{\int_0^\infty s^{\xi-\alpha-1}e^{-\frac{\delta}{2}s}\left\|\mathbb{S}_{\Psi_3}(t-s+\tau)-\mathbb{S}_{\Psi_3}(t-s)\right\|ds\right\}^p\\
&\le c(\alpha)^p\,C_2(\Gamma,\alpha,\delta,\xi,p)\,\sup_t \mathbf{E}\left\|\mathbb{S}_{\Psi_3}(t+\tau)-\mathbb{S}_{\Psi_3}(t)\right\|^p\\
&< c(\alpha)^p\,C_2(\Gamma,\alpha,\delta,\xi,p)C_pC_1(\Gamma,\xi,\delta,p)\eta\,.
\end{aligned}
$$

For L_2'', using Lemma 5.1 it follows that

$$
\begin{aligned}
L_2'' &\le \mathbf{E}\left\{\int_\varepsilon^\infty s^{\xi-1}\left\|\left[U(t+\tau,t-s+\tau)P(t-s+\tau)-U(t,t-s)P(t-s)\right]\mathbb{S}_{\Psi_3}(t-s)\right\|_\alpha ds\right\}^p\\
&\le C_2(\Gamma,\alpha,\delta,\xi,p)\,\varepsilon^p\sup_{t\in\mathbb{R}}\mathbf{E}\left\|\mathbb{S}_{\Psi_3}(t)\right\|^p\\
&\le C_2(\Gamma,\alpha,\delta,\xi,p)\,\varepsilon^pC_p\,C_1(\Gamma,\xi,\delta,p)\sup_{s\in\mathbb{R}}\mathbf{E}\left\|\Psi_3(s)\right\|_{\mathbb{L}_2^0}^p.
\end{aligned}
$$

The evaluation of the last term is straightforward. We obtain

$$
\begin{aligned}
L_3'' &\le \mathbf{E}\left\{\int_0^\varepsilon s^{\xi-1}\left\|[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)]\mathbb{S}_{\Psi_3}(t-s)\right\|_\alpha ds\right\}^p\\
&\le 2^p M^p\,\varepsilon^p\sup_{t\in\mathbb{R}}\mathbf{E}\left\|\mathbb{S}_{\Psi_3}(t)\right\|^p\\
&\le 2^p M^p\,\varepsilon^pC_p\,C_1(\Gamma,\xi,\delta,p)\sup_{s\in\mathbb{R}}\mathbf{E}\left\|\Psi_3(s)\right\|_{\mathbb{L}_2^0}^p.
\end{aligned}
$$

As to $p=2$, we have

$$
\begin{aligned}
L_1'' &= \mathbf{E}\Big\|\int_0^\infty \Gamma(t+\tau,t-s+\tau)\big[\Psi_3(t-s+\tau)-\Psi_3(t-s)\big]\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\le c(\alpha)^2\int_0^\infty s^{-2\alpha}e^{-\delta s}\mathbf{E}\big\|\Psi_3(t-s+\tau)-\Psi_3(t-s)\big\|^2\,ds\\
&\le c(\alpha)^2\Big(\int_0^\infty s^{-2\alpha}e^{-\delta s}\,ds\Big)\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi_3(t+\tau)-\Psi_3(t)\big\|^2\\
&\le c(\alpha)^2\frac{\Gamma(1-2\alpha)}{\delta^{1-2\alpha}}\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi_3(t+\tau)-\Psi_3(t)\big\|^2.
\end{aligned}
$$

For L_2'', using Lemma 5.1 it follows that

$$
\begin{aligned}
L_2'' &= \mathbf{E}\Big\|\int_\varepsilon^\infty \big[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_3(t-s)\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\le \frac{C(\alpha)}{\delta}\,\varepsilon^2\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi_3(t)\big\|^2.
\end{aligned}
$$

As to L_3'', it is straightforward. We obtain

$$
\begin{aligned}
L_3'' &= \mathbf{E}\Big\|\int_0^\varepsilon \big[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_3(t-s)\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\le 4C(\alpha)M^2\,\varepsilon\sup_{t\in\mathbb{R}}\mathbf{E}\big\|\Psi_3(t)\big\|^2.
\end{aligned}
$$

Combining these evaluations, we conclude that $\Gamma_5 X\in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$.

The proof for $\Gamma_6 X(\cdot)$ is similar to that of $\Gamma_5 X(\cdot)$ except that Eqs. (2.46) and (2.51) are used instead of Eqs. (2.45) and (2.52), respectively.

We are now ready to prove Theorem 6.3.

Proof. Consider the nonlinear operator $\mathbb{M}$ on the space $AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$ equipped with the α-sup norm $\|X\|_{\infty,\alpha}=\sup_{t\in\mathbb{R}}(\mathbf{E}\|X(t)\|_\alpha^p)^{1/p}$ and defined by

$$
\begin{aligned}
\mathbb{M}X(t) = &-F(t,BX(t))-\int_{-\infty}^t A(s)U(t,s)P(s)\,F_1(s,BX(s))\,ds\\
&+\int_t^\infty A(s)U(t,s)Q(s)\,F_1(s,BX(s))\,ds\\
&+\int_{-\infty}^t U(t,s)P(s)\,F_2(s,CX(s))\,ds-\int_t^\infty U(t,s)Q(s)\,F_2(s,CX(s))\,ds\\
&+\int_{-\infty}^t U(t,s)P(s)\,F_3(s,LX(s))\,d\mathbb{W}(s)-\int_t^\infty U(t,s)Q(s)\,F_3(s,LX(s))\,d\mathbb{W}(s)
\end{aligned}
$$

for each $t\in\mathbb{R}$.

As we have previously seen, for every $X\in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$, $F_1(\cdot,BX(\cdot))\in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\beta))\subset AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. In view of Lemmas 6.4, 6.5, and 6.6, it follows that $\mathbb{M}$ maps $AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$ into itself. To complete the proof one has to show that $\mathbb{M}$ has a unique fixed point.

Let $X, Y \in AP(\mathbb{R};L^p(\Omega;\mathbb{H}_\alpha))$. By $(5H)_1$, $(6H)_5$, and $(6H)_6$, we obtain

$$\begin{aligned}\mathbf{E}\big\|F_1(t,BX(t))-F_1(t,BY(t))\big\|_\alpha^p &\leq k(\alpha)\,K_1\mathbf{E}\big\|BX(t)-BY(t)\big\|^p\\ &\leq k(\alpha)\cdot K_1\,\varpi^p\,\big\|X-Y\big\|_{\infty,\alpha}^p.\end{aligned}$$

Now for Γ_1 and Γ_2, we have the following evaluations:

$$\begin{aligned}&\mathbf{E}\big\|(\Gamma_1X)(t)-(\Gamma_1Y)(t)\big\|_\alpha^p\\ &\quad\leq \mathbf{E}\left(\int_{-\infty}^t \big\|A(s)U(t,s)P(s)[F_1(s,BX(s))-F_1(s,BY(s))]\big\|_\alpha\,ds\right)^p\\ &\quad\leq n(\alpha,\beta)^p\left(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}\alpha}e^{-\frac{\delta}{2}(t-s)}\,ds\right)^{p-1}\times\\ &\quad\times\left(\int_{-\infty}^t (t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\mathbf{E}\big\|F_1(s,BX(s))-F_1(s,BY(s))\big\|_\beta^p\,ds\right)\\ &\quad\leq n(\alpha,\beta)^p\,k_1\,K_1\,C_2(\Gamma,\alpha,\delta,\xi,p)\varpi^p\big\|X-Y\big\|_{\infty,\alpha}^p.\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathbf{E}\big\|(\Gamma_2X)(t)-(\Gamma_2Y)(t)\big\|_\alpha^p\\ &\quad\leq \mathbf{E}\left(\int_t^\infty \big\|A(s)U(t,s)Q(s)[F_1(s,BX(s))-F_1(s,BY(s))]\big\|_\alpha\,ds\right)^p\\ &\quad\leq \frac{m(\alpha,\beta)^p\,k_1\,K_1}{\delta^p}\,\varpi^p\big\|X-Y\big\|_{\infty,\alpha}^p.\end{aligned}$$

As to Γ_3 and Γ_4, we have the following evaluations:

$$\begin{aligned}&\mathbf{E}\big\|(\Gamma_3X)(t)-(\Gamma_3Y)(t)\big\|_\alpha^p\\ &\quad\leq \mathbf{E}\left(\int_{-\infty}^t \big\|U(t,s)P(s)[F_2(s,CX(s))-F_2(s,CY(s))]\big\|_\alpha\,ds\right)^p\\ &\quad\leq c(\alpha)^p\left(\int_{-\infty}^t (t-s)^{-\frac{p}{p-1}\alpha}e^{-\frac{\delta}{2}(t-s)}\,ds\right)^{p-1}\times\\ &\quad\times\left(\int_{-\infty}^t (t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\mathbf{E}\big\|F_2(s,CX(s))-F_2(s,CY(s))\big\|_\alpha^p\,ds\right)\\ &\quad\leq k_2\,K_2\cdot c(\alpha)^p\,C_2(\Gamma,\alpha,\delta,\xi,p)\,\varpi^p\,\big\|X-Y\big\|_{\infty,\alpha}^p.\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\mathbf{E}\left\|(\Gamma_4 X)(t)-(\Gamma_4 Y)(t)\right\|_\alpha^p \\
&\leq \mathbf{E}\left(\int_t^\infty \left\|U(t,s)Q(s)[F_2(s,CX(s))-F_2(s,CY(s))]\right\|_\alpha ds\right)^p \\
&\leq \frac{k_2 K_2\, m(\alpha,\beta)^p}{\delta^p}\,\varpi^p \left\|X-Y\right\|_{\infty,\alpha}^p .
\end{aligned}
$$

As to Γ_5 and Γ_6, using the factorization method and subsequently invoking Proposition 3.26 leads to the estimate

$$
\begin{aligned}
&\mathbf{E}\left\|(\Gamma_5 X)(t)-(\Gamma_5 Y)(t)\right\|_\alpha^p \\
&\leq C_p M(\alpha)^p C_2(\Gamma,\alpha,\xi,\delta,p)\, C_1(\Gamma,\xi,\delta,p) \sup_{t\in\mathbb{R}} \mathbf{E}\left\|F(t,CX(t))-F(t,CY(t))\right\|^p \\
&\leq k_3\, C_p M(\alpha)^p K_3\, C_2(\Gamma,\alpha,\xi,\delta,p)\, C_1(\Gamma,\xi,\delta,p)\,\varpi^p \left\|X-Y\right\|_{\alpha,\infty}^p .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathbf{E}\left\|(\Gamma_6 X)(t)-(\Gamma_6 Y)(t)\right\|_\alpha^p \\
&\leq k_3 \,\frac{m(\alpha,\beta)^p}{\delta^p}\cdot C_p\cdot K_3\cdot C_1(\Gamma,\xi,\delta,p)\,\varpi^p \left\|X-Y\right\|_{\infty,\alpha}^p .
\end{aligned}
$$

For $p=2$,

$$
\begin{aligned}
&\mathbf{E}\left\|(\Gamma_5 X)(t)-(\Gamma_5 Y)(t)\right\|_\alpha^2 \\
&= \mathbf{E}\left\|\int_{-\infty}^t U(t,s)P(s)\left[F_3(s,LX(s))-F_3(s,LY(s))\right] d\mathbb{W}(s)\right\|_\alpha^2 \\
&\leq c(\alpha)^2 \int_{-\infty}^t (t-s)^{-2\alpha} e^{-\delta(t-s)} \mathbf{E}\left\|F_3(s,LX(s))-F_3(s,LY(s))\right\|_{\mathbb{L}_2^0}^2 ds \\
&\leq c^2(\alpha)\, k_3\, K_3\, \varpi^2 \Gamma\left(1-2\alpha)\delta^{2\alpha-1}\right) \left\|X-Y\right\|_{\infty,\alpha}^2
\end{aligned}
$$

which implies

$$
\left\|\Gamma_5 X-\Gamma_5 Y\right\|_{\infty,\alpha} \leq \cdot k_3'\cdot K(\alpha,\delta,\Gamma)\cdot K_3'\cdot\varpi \left\|X-Y\right\|_{\infty,\alpha}.
$$

Similarly,

$$
\begin{aligned}
&\mathbf{E}\left\|(\Gamma_6 X)(t)-(\Gamma_6 Y)(t)\right\|_\alpha^2 \\
&= \mathbf{E}\left\|\int_t^\infty U(t,s)Q(s)\left[F_3(s,LX(s))-F_3(s,LX(s))\right] d\mathbb{W}(s)\right\|_\alpha^2 \\
&\leq m(\alpha,\beta)^2 \int_t^\infty e^{\delta(t-s)} \mathbf{E}\left\|F_3(s,LX(s))-F_3(s,LX(s))\right\|_{\mathbb{L}_2^0}^2 \\
&\leq m(\alpha,\beta)^2\cdot k_3\cdot K_3\cdot\left(\frac{1}{\delta}\right)\varpi^2 \left\|X-Y\right\|_{\infty,\alpha}^2
\end{aligned}
$$

which implies

$$\left\|\Gamma_6 X - \Gamma_6 Y\right\|_{\infty,\alpha} \leq k_3' \cdot K_3' \cdot K(\alpha,\beta) \cdot \frac{1}{\sqrt{\delta}} \cdot \varpi \left\|X - Y\right\|_{\infty,\alpha}.$$

Consequently,

$$\left\|\mathbb{M}X - \mathbb{M}Y\right\|_{\infty,\alpha} \leq \Theta \cdot \left\|X - Y\right\|_{\infty,\alpha}.$$

Clearly, if $\Theta < 1$, then Eq. (6.6) has a unique fixed-point by the Banach fixed-point theorem, which is obviously the only p-th mean almost periodic solution to it.

6.3 Existence Results Through the Schauder Fixed Point Theorem

6.3.1 Existence of p-th Mean Almost Periodic Mild Solutions

In this section, we study the existence of p-th mean almost periodic solutions to the class of nonautonomous stochastic differential equations of type (5.3) where $(A(t))_{t\in\mathbb{R}}$ is a family of closed linear operators on $L^p(\Omega;\mathbb{H})$ satisfying Acquistapace–Terreni conditions (2.38)–(2.39), and the functions $F: \mathbb{R}\times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{H})$, $G: \mathbb{R}\times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{L}_2^0)$ are p-th mean almost periodic in $t\in\mathbb{R}$ uniformly in the second variable, and $\mathbb{W}(t)$ is a $\mathcal{Q}$-Wiener process taking its values in $\mathbb{K}$ with the real number line as time parameter.

Our method for investigating the existence of p-th mean almost periodic solutions to (5.3) relies heavily on ideas and techniques utilized in Goldstein–N'Guérékata [80] and Diagana [55] and the Schauder fixed point theorem.

To study the existence of p-th mean almost periodic solutions to Eq. (5.3), we suppose that the injection

$$\mathbb{H}_\alpha \hookrightarrow \mathbb{H}$$

is compact and in addition to $(5H)_3$ and $(6H)_6$, we require that the following assumptions hold:

$(6H)_9$ $R(\zeta, A(\cdot)) \in AP(L^p(\Omega;\mathbb{H}))$

$(6H)_{10}$ The function $F: \mathbb{R}\times L^p(\Omega,\mathbb{H}) \to L^p(\Omega,\mathbb{H})$ is p-th mean almost periodic in the first variable uniformly in the second variable. Furthermore, $X \to F(t,X)$ is uniformly continuous on any bounded subset $\mathcal{O}$ of $L^p(\Omega,\mathbb{H})$ for each $t\in\mathbb{R}$. Finally,

$$\sup_{t\in\mathbb{R}} \mathbf{E}\left\|F(t,X)\right\|^p \leq \mathcal{M}_1\left(\left\|X\right\|_\infty\right)$$

where $\mathcal{M}_1: \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r\to\infty} \frac{\mathcal{M}_1(r)}{r} = 0.$$

$(6H)_{11}$ The function $G:\mathbb{R}\times L^p(\Omega,\mathbb{H})\to L^p(\Omega,\mathbb{L}_2^0)$ is p-th mean almost periodic in the first variable uniformly in the second variable. Furthermore, $X\to G(t,X)$ is uniformly continuous on any bounded subset $\mathscr{O}'$ of $L^p(\Omega,\mathbb{H})$ for each $t\in\mathbb{R}$. Finally,

$$\sup_{t\in\mathbb{R}}\mathbf{E}\big\|G(t,X)\big\|^p\leq\mathscr{M}_2\big(\|X\|_\infty\big)$$

where $\mathscr{M}_2:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r\to\infty}\frac{\mathscr{M}_2(r)}{r}=0\,.$$

Remark 6.1. Let us mention that the fact the injection $\mathbb{H}_\alpha\hookrightarrow\mathbb{H}$ is compact yields that the injection

$$L^p(\Omega,\mathbb{H}_\alpha)\hookrightarrow L^p(\Omega,\mathbb{H})$$

is compact, too.

In this section, Γ_1 and Γ_2 stand respectively for the nonlinear integral operators defined by

$$(\Gamma_1X)(t):=\int_{-\infty}^t U(t,s)F(s,X(s))\,ds \text{ and } (\Gamma_2X)(t):=\int_{-\infty}^t U(t,s)G(s,X(s))\,d\mathbb{W}(s)\,.$$

Throughout this section we assume that $\alpha\in\big(0,\frac{1}{2}-\frac{1}{p}\big)$ if $p>2$ and $\alpha\in\big(0,\frac{1}{2}\big)$ if $p=2$. Moreover, we suppose that

$$2\beta>\alpha+1.$$

Lemma 6.7. *Under assumptions $(5H)_3$, $(6H)_6$, $(6H)_9$, $(6H)_{10}$, and $(6H)_{11}$, the mappings $\Gamma_i(i=1,2):BC(\mathbb{R},L^p(\Omega,\mathbb{H}))\to BC(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha))$ are well defined and continuous.*

Proof. We first show that $\Gamma_i\big(BC\big(\mathbb{R},L^p(\Omega,\mathbb{H})\big)\big)\subset BC\big(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha)\big)$ $(i=1,2)$. Let us start with Γ_1X. Indeed, using (2.47) it follows that for all $X\in BC(\mathbb{R},L^p(\Omega,\mathbb{H}))$,

$$\begin{aligned}
&\mathbf{E}\big\|\Gamma_1X(t)\big\|_\alpha^p\\
&\leq\mathbf{E}\Big[\int_{-\infty}^t c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\big\|F(s,X(s))\big\|\,ds\Big]^p\\
&\leq c(\alpha)^p\Big(\int_{-\infty}^t(t-s)^{-\frac{p}{p-1}\alpha}e^{-\frac{\delta}{2}(t-s)}\,ds\Big)^{p-1}\Big(\int_{-\infty}^t e^{-\frac{\delta}{2}(t-s)}\mathbf{E}\big\|F(s,X(s))\big\|^p\,ds\Big)\\
&\leq c(\alpha)^p\Big(\Gamma\big(1-\frac{p}{p-1}\alpha\big)\Big(\frac{2}{\delta}\Big)^{1-\frac{p}{p-1}\alpha}\Big(\frac{2}{\delta}\Big)\Big)^{p-1}\mathscr{M}_1\big(\|X\|_\infty\big)\\
&\leq c(\alpha)^p\Big(\Gamma\big(1-\frac{p}{p-1}\alpha\big)\Big)^{p-1}\Big(\frac{2}{\delta}\Big)^{p(1-\alpha)}\mathscr{M}_1\big(\|X\|_\infty\big)\,,
\end{aligned}$$

and hence

$$\left\|\Gamma_1 X\right\|^p_{\alpha,\infty} := \sup_{t\in\mathbb{R}} \mathbf{E}\left\|\Gamma_1 X(t)\right\|^p_\alpha \le l(\alpha,\delta,p)\mathscr{M}_1\left(\left\|X\right\|_\infty\right),$$

where $l(\alpha,\delta,p) = c(\alpha)^p\Big(\Gamma\big(1-\frac{p}{p-1}\alpha\big)\Big)^{p-1}\Big(\frac{2}{\delta}\Big)^{p(1-\alpha)}$.

As to $\Gamma_2 X$, we proceed in two steps. For $p>2$, we use the estimates obtained in Proposition 2.16 (ii) to get

$$\begin{aligned}\mathbf{E}\left\|\Gamma_2 X(t)\right\|^p_\alpha &\le k(\alpha,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\left\|G(s,X(s))\right\|^p_{\mathbb{L}_2^0}\\ &\le k(\alpha,\xi,\delta,p)\mathscr{M}_2\left(\left\|X\right\|_\infty\right),\end{aligned}$$

and hence

$$\left\|\Gamma_2 X\right\|^p_{\alpha,\infty} \le k(\alpha,\xi,\delta,p)\mathscr{M}_2\left(\left\|X\right\|_\infty\right),$$

where $k(\alpha,\xi,\delta,p)$ is a positive constant.

For $p=2$, we have

$$\begin{aligned}\mathbf{E}\left\|\Gamma_2 X(t)\right\|^2_\alpha &= \mathbf{E}\left\|\int_{-\infty}^t U(t,s)G(s,X(s))\,d\mathbb{W}(s)\right\|^2_\alpha\\ &\le c(\alpha)^2\int_{-\infty}^t (t-s)^{-2\alpha}e^{-\delta(t-s)}\mathbf{E}\left\|G(s,X(s))\right\|^2_{\mathbb{L}_2^0}\\ &\le c(\alpha)^2\Gamma\big(1-2\alpha\big)\delta^{1-2\alpha}\mathscr{M}_2\left(\left\|X\right\|_\infty\right),\end{aligned}$$

and hence

$$\left\|\Gamma_2 X\right\|^2_{\alpha,\infty} \le s(\alpha,\delta)\mathscr{M}_2\left(\left\|X\right\|_\infty\right),$$

where $s(\alpha,\delta) = c(\alpha)^2\Gamma\big(1-2\alpha\big)\delta^{1-2\alpha}$.

For the continuity, let $X^n \in AP(\mathbb{R};L^p(\Omega,\mathbb{H}))$ be a sequence which converges to some $X \in AP(\mathbb{R};L^p(\Omega,\mathbb{H}))$, that is, $\|X^n - X\|_\infty \to 0$ as $n\to\infty$. It follows from the estimates in Lemma 2.45 that

$$\begin{aligned}&\mathbf{E}\left\|\int_{-\infty}^t U(t,s)[F(s,X^n(s))-F(s,X(s))]\,ds\right\|^p_\alpha\\ &\quad\le \mathbf{E}\Big[\int_{-\infty}^t c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\left\|F(s,X^n(s))-F(s,X(s))\right\|ds\Big]^p.\end{aligned}$$

Now, using the continuity of F and the Lebesgue Dominated Convergence Theorem we obtain that

$$\mathbf{E}\left\|\int_{-\infty}^t U(t,s)[F(s,X^n(s))-F(s,X(s))]\,ds\right\|^p_\alpha \to 0 \text{ as } n\to\infty.$$

Therefore,

$$\left\|\Gamma_1 X^n - \Gamma_1 X\right\|_{\infty,\alpha} \to 0 \text{ as } n\to\infty.$$

For the term containing the Wiener process $\mathbb{W}$, we use the estimates (2.45) and (2.52) to obtain

$$\mathbf{E}\left\|\int_{-\infty}^{t} U(t,s)[G(s,X^n(s)) - G(s,X(s))]\,d\mathbb{W}(s)\right\|_{\alpha}^{p}$$
$$\leq k(\alpha,\xi,\delta,p)\sup_{t\in\mathbb{R}}\mathbf{E}\big\|G(t,X^n(t)) - G(t,X(t))\big\|^{p}$$

for $p > 2$ and

$$\mathbf{E}\left\|\int_{-\infty}^{t} U(t,s)[G(s,X^n(s)) - G(s,X(s))]\,d\mathbb{W}(s)\right\|_{\alpha}^{2}$$
$$\leq n(\alpha)^2\int_{-\infty}^{t}(t-s)^{-2\alpha}e^{-\delta\,(t-s)}\mathbf{E}\big\|G(s,X(s)^n) - G(s,X(s))\big\|^{2}\,ds$$

for $p = 2$.

Now, using the continuity of G and the Lebesgue Dominated Convergence Theorem we obtain that

$$\mathbf{E}\left\|\int_{-\infty}^{t} U(t,s)[G(s,X^n(s)) - G(s,X(s))]\,d\mathbb{W}(s)\right\|_{\alpha}^{p} \to 0 \text{ as } n \to \infty\,.$$

Therefore,

$$\big\|\Gamma_2 X^n - \Gamma_2 X\big\|_{\infty,\alpha} \to 0 \text{ as } n \to \infty.$$

Lemma 6.8. *Under assumptions* $(5H)_3$, $(6H)_6$, $(6H)_9$, $(6H)_{10}$, *and* $(6H)_{11}$, *the integral operator* $\Gamma_i(i = 1,2)$ *maps* $AP\big(\mathbb{R}, L^p(\Omega,\mathbb{H})\big)$ *into itself.*

Proof. Let us first show that $\Gamma_1 X(\cdot)$ is p-th mean almost periodic and let $f(t) = F(t,X(t))$. Indeed, assuming that X is p-th mean almost periodic and using assumption $(6H)_{10}$, Theorem 4.5, and Lemma 5.1, given $\varepsilon > 0$, one can find $l_\varepsilon > 0$ such that any interval of length l_ε contains at least τ with the property that

$$\big\|U(t+\tau,s+\tau) - U(t,s)\big\| \leq \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \geq \varepsilon$, and

$$\mathbf{E}\big\|f(\sigma+\tau) - f(\sigma)\big\|^{p} < \eta$$

for each $\sigma \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Moreover, it follows from Lemma 4.1 (ii) that there exists a positive constant K_1 such that

$$\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|f(\sigma)\big\|^{p} \leq K_1\,.$$

Now, using assumption $(5H)_3$ and Hölder's inequality, we obtain

$$\mathbf{E}\big\|f(t+\tau) - f(t)\big\|^{p}$$
$$\leq 3^{p-1}\,\mathbf{E}\Big[\int_{0}^{\infty}\big\|U(t+\tau,t+\tau-s)\big\|\,\big\|f(t+\tau-s) - f(t-s)\big\|\,ds\Big]^{p}$$
$$+3^{p-1}\,\mathbf{E}\Big[\int_{\varepsilon}^{\infty}\big\|U(t+\tau,t+\tau-s) - U(t,t-s)\big\|\,\big\|f(t-s)\big\|\,ds\Big]^{p}$$

$$+3^{p-1}\,\mathbf{E}\Big[\int_0^{\varepsilon}\big\|U(t+\tau,t+\tau-s)-U(t,t-s)\big\|\,\big\|f(t-s)\big\|\,ds\Big]^p$$

$$\leq 3^{p-1}M^p\,\mathbf{E}\Big[\int_0^{\infty}e^{-\delta s}\,\big\|f(t+\tau-s)-f(t-s)\big\|\,ds\Big]^p$$

$$+3^{p-1}\,\varepsilon^p\,\mathbf{E}\Big[\int_{\varepsilon}^{\infty}e^{-\frac{\delta}{2}s}\,\big\|f(t-s)\big\|\,ds\Big]^p+3^{p-1}M^p\,\mathbf{E}\Big[\int_0^{\varepsilon}2e^{-\delta s}\,\big\|f(t-s)\big\|\,ds\Big]^p$$

$$\leq 3^{p-1}M^p\Big(\int_0^{\infty}e^{-\delta s}\,ds\Big)^{p-1}\Big(\int_0^{\infty}e^{-\delta s}\,\mathbf{E}\big\|f(t+\tau-s)-f(t-s)\big\|^p\,ds\Big)$$

$$+3^{p-1}\,\varepsilon^p\Big(\int_0^{\infty}e^{-\delta s}\,ds\Big)^{p-1}\Big(\int_{\varepsilon}^{\infty}e^{-\frac{\delta ps}{2}}\,\mathbf{E}\big\|f(t-s)\big\|^p\,ds\Big)$$

$$+6^{p-1}M^p\Big(\int_0^{\varepsilon}e^{-\delta s}\,ds\Big)^{p-1}\Big(\int_0^{\varepsilon}e^{-\frac{\delta ps}{2}}\,\mathbf{E}\big\|f(t-s)\big\|^p\,ds\Big)$$

$$\leq 3^{p-1}M^p\Big(\int_0^{\infty}e^{-\delta s}\,ds\Big)^p\sup_{s\in\mathbb{R}}\mathbf{E}\big\|f(t+\tau-s)-f(t-s)\big\|^p$$

$$+3^{p-1}\,\varepsilon^p\Big(\int_{\varepsilon}^{\infty}e^{-\delta s}\,ds\Big)^p\sup_{s\in\mathbb{R}}\mathbf{E}\big\|f(t-s)\big\|^p$$

$$+6^{p-1}M^p\Big(\int_0^{\varepsilon}e^{-\delta s}\,ds\Big)^p\sup_{s\in\mathbb{R}}\mathbf{E}\big\|f(t-s)\big\|^p$$

$$\leq 3^{p-1}M^p\Big(\frac{1}{\delta^p}\Big)\eta+3^{p-1}M^pK_1\Big(\frac{1}{\delta^p}\Big)\varepsilon^p+6^{p-1}M^p\varepsilon^pK_1\,\varepsilon^p.$$

As to $\Gamma_2X(\cdot)$, we split the proof in two cases: $p>2$ and $p=2$. To this end, we let $g(t)=G(t,X(t))$. Let us start with the case where $p>2$. Assuming that X is p-th mean almost periodic and using assumption $(6H)_{11}$, Theorem 4.5, and Lemma 5.1, given $\varepsilon>0$, one can find $l_\varepsilon>0$ such that any interval of length l_ε contains at least τ with the property that

$$\big\|U(t+\tau,s+\tau)-U(t,s)\big\|\leq\varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t-s\geq\varepsilon$, and

$$\mathbf{E}\big\|g(\sigma+\tau)-g(\sigma)\big\|^p<\eta$$

for each $\sigma\in\mathbb{R}$, where $\eta(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Moreover, it follows from Lemma 4.1 (ii) that there exists a positive constant K_2 such that

$$\sup_{\sigma\in\mathbb{R}}\mathbf{E}\big\|g(\sigma)\big\|^p\leq K_2\,.$$

Now

$$\mathbf{E}\big\|g(t+\tau)-g(t)\big\|^p$$

$$\leq 3^{p-1}\,\mathbf{E}\left\|\int_0^\infty U(t+\tau,t+\tau-s)\left[g(t+\tau-s)-g(t-s)\right]d\mathbb{W}(s)\right\|^p$$

$$+3^{p-1}\,\mathbf{E}\left\|\int_\varepsilon^\infty \left[U(t+\tau,t+\tau-s)-U(t,t-s)\right]g(t-s)\,d\mathbb{W}(s)\right\|^p$$

$$+3^{p-1}\,\mathbf{E}\left\|\int_0^\varepsilon \left[U(t+\tau,t+\tau-s)-U(t,t-s)\right]g(t-s)\,d\mathbb{W}(s)\right\|^p.$$

The next step consists in proving the p-th mean almost periodicity of $\Gamma_2 X(\cdot)$. Using assumption $(5H)_3$, Hölder's inequality, and Proposition 3.23, we have

$$\mathbf{E}\left\|g(t+\tau)-g(t)\right\|^p$$

$$\leq 3^{p-1}\,C_p\,\mathbf{E}\Big[\int_0^\infty \left\|U(t+\tau,t+\tau-s)\right\|^2\left\|g(t+\tau-s)-g(t-s)\right\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}$$

$$+3^{p-1}\,C_p\,\mathbf{E}\Big[\int_\varepsilon^\infty \left\|U(t+\tau,t+\tau-s)-U(t,t-s)\right\|^2\left\|g(t-s)\right\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}$$

$$+3^{p-1}\,C_p\,\mathbf{E}\Big[\int_0^\varepsilon \left\|U(t+\tau,t+\tau-s)-U(t,t-s)\right\|^2\left\|g(t-s)\right\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}$$

$$\leq 3^{p-1}\,C_p\,M^p\,\mathbf{E}\Big[\int_0^\infty e^{-2\delta s}\left\|g(t+\tau-s)-g(t-s)\right\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}$$

$$+3^{p-1}\,C_p\,\varepsilon^p\mathbf{E}\Big[\int_\varepsilon^\infty e^{-\delta s}\left\|g(t-s)\right\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}$$

$$+3^{p-1}\,2^{p/2}\,C_p\,\mathbf{E}\Big[\int_0^\varepsilon e^{-2\delta s}\left\|g(t-s)\right\|^2_{\mathbb{L}_2^0}\,ds\Big]^{p/2}$$

$$\leq 3^{p-1}\,C_p\,M^p\left(\int_0^\infty e^{-\frac{p\,\delta\,s}{p-2}}\,ds\right)^{\frac{p-2}{2}}\left(\int_0^\infty e^{-\frac{p\,\delta\,s}{2}}\left\|g(t+\tau-s)-g(t-s)\right\|^p_{\mathbb{L}_2^0}\,ds\right)$$

$$+3^{p-1}\,C_p\,\varepsilon^p\left(\int_\varepsilon^\infty e^{-\frac{p\,\delta\,s}{2(p-2)}}\,ds\right)^{\frac{p-2}{2}}\left(\int_\varepsilon^\infty e^{-\frac{p\,\delta\,s}{4}}\,\mathbf{E}\left\|g(t-s)\right\|^p_{\mathbb{L}_2^0}\,ds\right)$$

$$+3^{p-1}\,2^{p/2}\,C_p\,M^p\left(\int_0^\varepsilon e^{-\frac{p\,\delta\,s}{p-2}}\,ds\right)^{\frac{p-2}{2}}\left(\int_0^\varepsilon e^{-\frac{p\,\delta\,s}{2}}\,\mathbf{E}\left\|g(t-s)\right\|^p_{\mathbb{L}_2^0}\,ds\right)$$

$$\leq 3^{p-1}\,C_p\,M^p\,\eta\left(\int_0^\infty e^{-\frac{p\,\delta\,s}{p-2}}\,ds\right)^{\frac{p-2}{2}}\left(\int_0^\infty e^{-\frac{p\,\delta\,s}{2}}\,ds\right)$$

$$+3^{p-1}\,C_p\,\varepsilon^p\,K_2\left(\int_\varepsilon^\infty e^{-\frac{p\,\delta\,s}{2(p-2)}}\,ds\right)^{\frac{p-2}{2}}\left(\int_\varepsilon^\infty e^{-\frac{p\,\delta\,s}{4}}\,ds\right)$$

$$+3^{p-1}\,2^{p/2}\,C_p\,M^p\,K_2\left(\int_0^\varepsilon e^{-\frac{p\,\delta\,s}{p-2}}\,ds\right)^{\frac{p-2}{2}}\left(\int_0^\varepsilon e^{-\frac{p\,\delta\,s}{2}}\,ds\right)$$

$$\leq 3^{p-1} C_p M^p \eta \left(\frac{p-2}{p\delta}\right)^{p-2} \left(\frac{2}{p\delta}\right)$$

$$+3^{p-1} C_p \varepsilon^p K_2 \left(\frac{2(p-2)}{p\delta}\right)^{\frac{p-2}{2}} \left(\frac{4}{p\delta}\right) + 3^{p-1} 2^{p/2} C_p M^p K_2 \varepsilon^p.$$

As to the case $p = 2$, we proceed in the same way using isometry inequality to obtain

$$\mathbf{E}\left\|\Gamma_2 X(t+\tau) - \Gamma_2 X(t)\right\|^2$$

$$\leq 3 M^2 \left(\int_0^\infty e^{-2\delta s}\, ds\right) \sup_{\sigma\in\mathbb{R}} \mathbf{E}\left\|g(\sigma+\tau) - g(\sigma)\right\|^2_{\mathbb{L}_2^0}$$

$$+3\,\varepsilon^2 \left(\int_\varepsilon^\infty e^{-\delta s}\, ds\right) \sup_{\sigma\in\mathbb{R}} \mathbf{E}\left\|g(\sigma)\right\|^2_{\mathbb{L}_2^0} + 6 M^2 \left(\int_0^\varepsilon e^{-2\delta s}\, ds\right) \sup_{\sigma\in\mathbb{R}} \mathbf{E}\left\|g(\sigma)\right\|^2_{\mathbb{L}_2^0}$$

$$\leq 3\left[\eta \frac{M^2}{2\delta} + \varepsilon \frac{K_2}{\delta} + 2\,\varepsilon K_2\right].$$

Hence, $\Gamma_2 X(\cdot)$ is p-th mean almost periodic.

Let $0 < \gamma \leq 1$ and set

$$BC^\gamma\big(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha)\big) = \Big\{X \in BC\big(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha)\big) : \left\|X\right\|_{\alpha,\gamma} < \infty\Big\},$$

where

$$\left\|X\right\|_{\alpha,\gamma} = \sup_{t\in\mathbb{R}} \Big[\mathbf{E}\left\|X(t)\right\|^p_\alpha\Big]^{\frac{1}{p}} + \gamma \sup_{t,\, s\in\mathbb{R},\, s\neq t} \frac{\Big[\mathbf{E}\left\|X(t) - X(s)\right\|^p_\alpha\Big]^{\frac{1}{p}}}{\left|t-s\right|^\gamma}.$$

Clearly, the space $BC^\gamma\big(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha)\big)$ equipped with the norm $\left\|\cdot\right\|_{\alpha,\gamma}$ is a Banach space, which is in fact the Banach space of all bounded continuous Hölder functions from $\mathbb{R}$ to $L^p(\Omega, \mathbb{H}_\alpha)$ whose Hölder exponent is γ.

Lemma 6.9. *Under assumptions $(5H)_3$, $(6H)_6$, $(6H)_9$, $(6H)_{10}$, $(6H)_{11}$, the mapping Γ_1 defined previously maps bounded sets of $BC\big(\mathbb{R}, L^p(\Omega, \mathbb{H})\big)$ into bounded sets of $BC^\gamma(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha))$ for some $0 < \gamma < 1$.*

Proof. Let $X \in BC(\mathbb{R}, L^p(\Omega, \mathbb{H}))$ and let $f(t) = F(t, X(t))$ for each $t \in \mathbb{R}$. Proceeding as before, we have

$$\begin{aligned}\mathbf{E}\left\|\Gamma_1 X(t)\right\|^p_\alpha &\leq c\mathbf{E}\left\|\Gamma_1 X(t)\right\|^p_\beta \\ &\leq c \cdot l(\beta, \delta, p) \mathscr{M}_1\left(\left\|X\right\|_\infty\right).\end{aligned}$$

Let $t_1 < t_2$. Clearly, we have

$$
\begin{aligned}
&\mathbf{E}\big\|(\Gamma_1 X)(t_2)-(\Gamma_1 X)(t_1)\big\|_\alpha^p \\
&\leq 2^{p-1}\mathbf{E}\left\|\int_{t_1}^{t_2} U(t_2,s)\,f(s)\,ds\right\|_\alpha^p + 2^{p-1}\mathbf{E}\left\|\int_{-\infty}^{t_1}[U(t_2,s)-U(t_1,s)]\,f(s)\,ds\right\|_\alpha^p \\
&= 2^{p-1}\mathbf{E}\left\|\int_{t_1}^{t_2} U(t_2,s)\,f(s)\,ds\right\|_\alpha^p + 2^{p-1}\mathbf{E}\left\|\int_{-\infty}^{t_1}\big(\int_{t_1}^{t_2}\frac{\partial U(\tau,s)}{\partial\tau}d\tau\big)\,f(s)\,ds\right\|_\alpha^p \\
&= 2^{p-1}\mathbf{E}\left\|\int_{t_1}^{t_2} U(t_2,s)\,f(s)\,ds\right\|_\alpha^p + 2^{p-1}\mathbf{E}\left\|\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}A(\tau)U(\tau,s)\,f(s)\,d\tau\Big)\,ds\right\|_\alpha^p \\
&= N_1+N_2.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
N_1 &\leq \mathbf{E}\Big\{\int_{t_1}^{t_2}\big\|U(t_2,s)\,f(s)\big\|_\alpha\,ds\Big\}^p \\
&\leq c(\alpha)^p\mathbf{E}\Big\{\int_{t_1}^{t_2}(t_2-s)^{-\alpha}e^{-\frac{\delta}{2}(t_2-s)}\big\|f(s)\big\|\,ds\Big\}^p \\
&\leq c(\alpha)^p\Big(\mathscr{M}_1\big(\|X\|_\infty\big)\Big)\Big(\int_{t_1}^{t_2}(t_2-s)^{-\frac{p}{p-1}\alpha}e^{-\frac{\delta}{2}(t_2-s)}\Big)^{p-1}\Big(\int_{t_1}^{t_2}e^{-\frac{\delta}{2}(t_2-s)}\,ds\Big) \\
&\leq c(\alpha)^p\Big(\mathscr{M}_1\big(\|X\|_\infty\big)\Big)\Big(\int_{t_1}^{t_2}(t_2-s)^{-\frac{p}{p-1}\alpha}\Big)^{p-1}\big(t_2-t_1\big) \\
&\leq c(\alpha)^p\mathscr{M}_1\big(\|X\|_\infty\big)\Big(1-\frac{p}{p-1}\alpha\Big)^{-(p-1)}(t_2-t_1)^{p(1-\alpha)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
N_2 &\leq \mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}\big\|A(\tau)U(\tau,s)\,f(s)\big\|_\alpha\,d\tau\Big)\,ds\Big\}^p \\
&\leq r(\alpha,\beta)^p\,\mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}(\tau-s)^{-\beta}e^{-\frac{\delta}{4}(\tau-s)}\big\|f(s)\big\|\,d\tau\Big)\,ds\Big\}^p \\
&\leq r(\alpha,\beta)^p\,\mathbf{E}\left[\int_{t_1}^{t_2}\Big(\int_{-\infty}^{t_1}(\tau-s)^{-\frac{p}{p-1}\beta}e^{-\frac{\delta}{4}(\tau-s)}ds\Big)^{\frac{p-1}{p}}\Big)\Big(\int_{-\infty}^{t_1}e^{-\frac{\delta}{4}(\tau-s)}\big\|f(s)\big\|^p ds\Big)^{\frac{1}{p}}d\tau\right]^p \\
&\leq r(\alpha,\beta)^p\Big(\int_{-\infty}^{t_1}e^{-\frac{\delta}{4}(t_1-s)}\mathbf{E}\big\|f(s)\big\|^p ds\Big)\Big[\int_{t_1}^{t_2}\Big(\int_{-\infty}^{t_1}(\tau-s)^{-\frac{p}{p-1}\beta}e^{-\frac{\delta}{4}(\tau-s)}ds\Big)^{\frac{p-1}{p}}d\tau\Big]^p \\
&\leq r(\alpha,\beta)^p\Big(\int_{-\infty}^{t_1}e^{-\frac{\delta}{4}(t_1-s)}\mathbf{E}\big\|f(s)\big\|^p ds\Big)\left[\int_{t_1}^{t_2}(\tau-t_1)^{-\beta}\Big(\int_{-\infty}^{t_1}e^{-\frac{\delta}{4}(\tau-s)}\,ds\Big)^{\frac{p-1}{p}}d\tau\right]^p \\
&\leq r(\alpha,\beta)^p\Big(\int_{-\infty}^{t_1}e^{-\frac{\delta}{4}(t_1-s)}\mathbf{E}\big\|f(s)\big\|^p ds\Big)\Big[\int_{t_1}^{t_2}(\tau-t_1)^{-\beta}\Big(\int_{\tau-t_1}^{\infty}e^{-\frac{\delta}{4}r}\,dr\Big)^{\frac{p-1}{p}}d\tau\Big]^p \\
&\leq r(\alpha,\beta)^p\mathscr{M}_1\big(\|X\|_\infty\big)\Big(\frac{4}{\delta}\Big)^p(1-\beta)^{-p}(t_2-t_1)^{p(1-\beta)}.
\end{aligned}
$$

For $\gamma=\min(1-\alpha,1-\beta)=1-\beta$, one has

$$\mathbf{E}\left\|(\Gamma_1X)(t_2)-(\Gamma_1X)(t_1)\right\|_\alpha^p \le s(\alpha,\beta,\delta)\mathscr{M}_1\left(\|X\|_\infty\right)\left|t_2-t_1\right|^{p\gamma}$$

where $s(\alpha,\beta,\delta)$ is a positive constant.

Lemma 6.10. *Let $0<\alpha<\beta<1/2$. Under assumptions $(5H)_3$, $(6H)_6$, $(6H)_9$, $(6H)_{10}$, $(6H)_{11}$, the mapping Γ_2 defined previously maps bounded sets of $BC\left(\mathbb{R},L^p(\Omega,\mathbb{H})\right)$ into bounded sets of $BC^\gamma(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha))$ for some $0<\gamma<1$.*

Proof. Let $X\in BC(\mathbb{R},L^p(\Omega,\mathbb{H}))$ and let $g(t)=G(t,X(t))$ for each $t\in\mathbb{R}$. We break down the computations in two cases: $p>2$ and $p=2$.

For $p>2$, we have

$$\begin{aligned}\mathbf{E}\left\|\Gamma_2X(t)\right\|_\alpha^p &\le c\mathbf{E}\left\|\Gamma_2X(t)\right\|_\beta^p\\ &\le c\cdot k(\beta,\xi,\delta,p)\,\mathscr{M}_2\left(\|X\|_\infty\right).\end{aligned}$$

Let $t_1<t_2$. Clearly,

$$\begin{aligned}&\mathbf{E}\left\|(\Gamma_2X)(t_2)-(\Gamma_2X)(t_1)\right\|_\alpha^p\\ &\le 2^{p-1}\mathbf{E}\left\|\int_{t_1}^{t_2}U(t_2,s)\,g(s)\,d\mathbb{W}(s)\right\|_\alpha^p\\ &\quad+2^{p-1}\mathbf{E}\left\|\int_{-\infty}^{t_1}[U(t_2,s)-U(t_1,s)]\,g(s)\,d\mathbb{W}(s)\right\|_\alpha^p\\ &= N_1'+N_2'.\end{aligned}$$

We use the factorization method (3.34) to obtain

$$\begin{aligned}N_1' &= \left|\frac{\sin(\pi\xi)}{\pi}\right|^p\mathbf{E}\left\|\int_{t_1}^{t_2}(t_2-s)^{\xi-1}U(t_2,s)\mathbb{S}_g(s)\,ds\right\|_\alpha^p\\ &\le \left|\frac{\sin(\pi\xi)}{\pi}\right|^p\mathbf{E}\Big[\int_{t_1}^{t_2}(t_2-s)^{\xi-1}\left\|U(t_2,s)\mathbb{S}_g(s)\right\|_\alpha ds\Big]^p\\ &\le M(\alpha)^p\left|\frac{\sin(\pi\xi)}{\pi}\right|^p\mathbf{E}\Big[\int_{t_1}^{t_2}(t_2-s)^{\xi-1}(t_2-s)^{\alpha}e^{-\frac{\delta}{2}(t_2-s)}\left\|\mathbb{S}_g(s)\right\|ds\Big]^p\\ &\le M(\alpha)^p\left|\frac{\sin(\pi\xi)}{\pi}\right|^p\Big(\int_{t_1}^{t_2}(t_2-s)^{-\frac{p}{p-1}\alpha}ds\Big)^{p-1}\times\\ &\qquad\times\Big(\int_{t_1}^{t_2}(t_2-s)^{-p(1-\xi)}e^{-p\frac{\delta}{2}(t_2-s)}\mathbf{E}\left\|\mathbb{S}_g(s)\right\|^p ds\Big)\\ &\le M(\alpha)^p\left|\frac{\sin(\pi\xi)}{\pi}\right|^p\Big(\int_{t_1}^{t_2}(t_2-s)^{-\frac{p}{p-1}\alpha}ds\Big)^{p-1}\times\\ &\qquad\times\Big(\int_{t_1}^{t_2}(t_2-s)^{-p(1-\xi)}e^{-p\frac{\delta}{2}(t_2-s)}ds\Big)\sup_{t\in\mathbb{R}}\mathbf{E}\left\|\mathbb{S}_g(t)\right\|^p\\ &\le s(\xi,\delta,\Gamma,p)\left(1-\frac{p}{p-1}\alpha\right)^{-(p-1)}\mathscr{M}_2\left(\|X\|_\infty\right)(t_2-t_1)^{p(1-\alpha)}\end{aligned}$$

where $s(\xi,\alpha,\delta,\Gamma,p)$ is a positive constant.

Similarly,

$$
\begin{aligned}
N_2' &= \mathbf{E}\left\|\int_{-\infty}^{t_1}\left[\int_{t_1}^{t_2}\frac{\partial}{\partial\tau}U(\tau,s)\,d\tau\right]g(s)\,d\mathbb{W}(s)\right\|_\alpha^p\\
&= \mathbf{E}\left\|\int_{-\infty}^{t_1}\left[\int_{t_1}^{t_2}A(\tau)U(\tau,s)\,d\tau\right]g(s)\,d\mathbb{W}(s)\right\|_\alpha^p.
\end{aligned}
$$

Now, using the representation (3.34) together with a stochastic version of the Fubini theorem (Proposition 3.24) gives us

$$
\begin{aligned}
N_2' &= \left|\frac{\sin(\pi\xi)}{\pi}\right|^p\mathbf{E}\left\|\int_{t_1}^{t_2}\Big(A(\tau)U(\tau,t_1)\int_{-\infty}^{t_1}(t_1-s)^{\xi-1}U(t_1,s)\mathbb{S}_g(s)\,ds\Big)\,d\tau\right\|_\alpha^p\\
&\le \left|\frac{\sin(\pi\xi)}{\pi}\right|^p\mathbf{E}\Big[\int_{t_1}^{t_2}\Big(\int_{-\infty}^{t_1}(t_1-s)^{\xi-1}\big\|A(\tau)U(\tau,s)\mathbb{S}_g(s)\big\|_\alpha\,ds\Big)\,d\tau\Big]^p\\
&\le r(\alpha,\beta)\left|\frac{\sin(\pi\xi)}{\pi}\right|^p\mathbf{E}\Big[\int_{t_1}^{t_2}\Big(\int_{-\infty}^{t_1}(t_1-s)^{\xi-1}(\tau-s)^{-\beta}e^{\frac{\delta}{4}(\tau-s)}\big\|\mathbb{S}_g(s)\big\|\,ds\Big)\,d\tau\Big]^p
\end{aligned}
$$

with ξ satisfying $\beta+\frac{1}{p}<\xi<\frac{1}{2}$.
Since $\tau>t_1$, it follows from Hölder's inequality that

$$
\begin{aligned}
N_2' &\le r(\alpha,\beta)\left|\frac{\sin(\pi\xi)}{\pi}\right|^p\mathbf{E}\Big[\int_{t_1}^{t_2}(\tau-t_1)^{-\beta}\Big(\int_{-\infty}^{t_1}(t_1-s)^{\xi-1}e^{-\frac{\delta}{4}(\tau-s)}\big\|\mathbb{S}_g(s)\big\|\,ds\Big)\,d\tau\Big]^p\\
&\le r(\alpha,\beta)\left|\frac{\sin(\pi\xi)}{\pi}\right|^p\mathbf{E}\Big[\Big(\int_{t_1}^{t_2}(\tau-t_1)^{-\beta}\,d\tau\Big)^p\Big(\int_{-\infty}^{t_1}(t_1-s)^{\xi-1}e^{-\frac{\delta}{4}(t_1-s)}\big\|\mathbb{S}_g(s)\big\|\,ds\Big)^p\Big]\\
&\le r(\alpha,\beta)\left|\frac{\sin(\pi\xi)}{\pi}\right|^p(t_2-t_1)^{p(1-\beta)}\Big(\int_{-\infty}^{t_1}(t_1-s)^{\frac{p}{p-1}(\xi-\alpha-1)}e^{\frac{\delta}{4}(t_1-s)}\,ds\Big)^{p-1}\times\\
&\quad\times\Big(\int_{-\infty}^{t_1}e^{-\frac{\delta}{4}(t_1-s)}\,ds\Big)\sup_{s\in\mathbb{R}}\mathbf{E}\big\|\mathbb{S}_g(s)\big\|^p\\
&\le r(\xi,\beta,\delta,\Gamma,p)\,(1-\beta)^{-p}\mathscr{M}_2\big(\|X\|_\infty\big)(t_2-t_1)^{p(1-\beta)}.
\end{aligned}
$$

For $\gamma=\min(1-\alpha,1-\beta)=1-\beta$, one has

$$
\Big[\mathbf{E}\Big\|(\Gamma_2X)(t_2)-(\Gamma_2X)(t_1)\Big\|_\alpha^p\Big]^{1/p}\le r(\xi,\beta,\delta,\Gamma,p)\,(1-\beta)^{-1}\Big[\mathscr{M}_2\big(\|X\|_\infty\big)\Big]^{1/p}(t_2-t_1)^\gamma.
$$

As to $p=2$, we have

$$
\begin{aligned}
\mathbf{E}\Big\|\Gamma_2X(t)\Big\|_\alpha^2 &\le c\,\mathbf{E}\Big\|\Gamma_2X(t)\Big\|_\beta^2\\
&\le c\cdot s(\beta,\delta)\mathscr{M}_2\big(\|X\|_\infty\big).
\end{aligned}
$$

For $t_1<t_2$, let us start with the first term. By Itô isometry identity, we have

$$
\begin{aligned}
N_1' &\leq c(\alpha)^2 \int_{t_1}^{t_2} (t_2-s)^{-2\alpha} e^{-\delta(t_2-s)} \mathbf{E}\big\|g(s)\big\|_{\mathbb{L}_2^0}^2 \, ds \\
&\leq c(\alpha)^2 \Big(\int_{t_1}^{t_2} (t_2-s)^{-2\alpha} \, ds \Big) \sup_{s\in\mathbb{R}} \mathbf{E}\big\|g(s)\big\|_{\mathbb{L}_2^0}^2 \\
&\leq c(\alpha)\,(1-2\alpha)^{-1} \mathscr{M}_2\big(\|X\|_\infty\big)\,(t_2-t_1)^{1-2\alpha}\,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
N_2' &= \mathbf{E}\Big\| \int_{-\infty}^{t_1} \Big[\int_{t_1}^{t_2} \frac{\partial}{\partial \tau} U(\tau,s)\, d\tau \Big] g(s)\, d\mathbb{W}(s) \Big\|_\alpha^2 \\
&= \mathbf{E}\Big\| \int_{-\infty}^{t_1} \Big[\int_{t_1}^{t_2} A(\tau) U(\tau,s)\, d\tau \Big] g(s)\, d\mathbb{W}(s) \Big\|_\alpha^2 \\
&= \mathbf{E}\Big\| \int_{t_1}^{t_2} A(\tau) U(\tau,t_1) \Big\{ \int_{-\infty}^{t_1} U(t_1,s)\, g(s)\, d\mathbb{W}(s) \Big\}\, d\tau \Big\|_\alpha^2 \\
&\leq \mathbf{E}\Big[\int_{t_1}^{t_2} \| \int_{-\infty}^{t_1} A(\tau) U(\tau,s)\, g(s)\, d\mathbb{W}(s) \|_\alpha^2 \, d\tau \Big]^2 \\
&\leq r(\alpha,\beta)^2 (t_2-t_1) \int_{t_1}^{t_2} \Big\{ \int_{-\infty}^{t_1} (\tau-s)^{-2\beta} e^{-\frac{\delta}{2}(\tau-s)} \mathbf{E}\big\|g(s)\big\|_{\mathbb{L}_2^0}^2 \, ds \Big\}\, d\tau \\
&\leq r(\alpha,\beta)^2 (t_2-t_1) \Big(\int_{t_1}^{t_2} (\tau-t_1)^{-2\beta}\, d\tau \Big) \Big(\int_{-\infty}^{t_1} e^{-\frac{\delta}{2}(t_1-s)} \mathbf{E}\big\|g(s)\big\|_{\mathbb{L}_2^0}^2 \, ds \Big) \\
&\leq r(\alpha,\beta)^2\,(1-2\beta)^{-1} \mathscr{M}_2\big(\|X\|_\infty\big)\,(t_2-t_1)^{2(1-\beta)}\,.
\end{aligned}
$$

For $\gamma = \min\big(\frac{1}{2}-\alpha, 1-\beta\big) = \frac{1}{2}-\alpha$ (since α, $\beta \in \big(0,\frac{1}{2}\big)$), one has

$$
\Big[\mathbf{E}\big\|(\Gamma_2 X)(t_2)-(\Gamma_2 X)(t_1)\big\|_\alpha^2\Big]^{1/2} \leq r(\xi,\beta,\delta)\,(1-2\beta)^{-1/2}\Big[\mathscr{M}_2\big(\|X\|_\infty\big)\Big]^{1/2}(t_2-t_1)^\gamma.
$$

Therefore, for each $X \in BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ such that $\mathbf{E}\big\|X(t)\big\|^p \leq R$ for all $t \in \mathbb{R}$, then $\Gamma_i X(t)$ belongs to $BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ with $\mathbf{E}\big\|\Gamma_i X(t)\big\|^p \leq R'$ where R' depends on R.

Lemma 6.11. *The integral operators $\Gamma_i (i=1,2)$ map bounded sets of $AP(\Omega, L^p(\Omega,\mathbb{H}))$ into bounded sets of $BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)) \cap AP(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ for $0<\gamma<\alpha$.*

Proof. The proof follows along the same lines as that of Lemma 6.9 and hence is omitted.

Similarly, the next lemma is a consequence of [80, Proposition 3.3]. Note in this context that $\mathbb{X} = L^p(\Omega,\mathbb{H})$ and $\mathbb{Y} = L^p(\Omega,\mathbb{H}_\alpha)$.

Lemma 6.12. *For $0<\gamma<\alpha$, the Banach space $BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ is compactly contained in $BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$, that is, the canonical injection*

$$
id:\ BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)) \to BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))
$$

is compact, which yields

$$id : BC^{\gamma}(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha)) \cap AP(\mathbb{R}, L^p(\Omega, \mathbb{H})) \to AP(\mathbb{R}, L^p(\Omega, \mathbb{H}))$$

is compact, too.

Theorem 6.4. *Suppose assumptions* $(5H)_3$, $(6H)_6$, $(6H)_9$, $(6H)_{10}$, *and* $(6H)_{11}$ *hold, then the nonautonomous differential equation* (5.3) *has at least one p-th mean almost periodic mild solution.*

Proof. Let us recall that in view of Lemmas 6.7 and 6.8, we have

$$\left\|(\Gamma_1+\Gamma_2)X\right\|_{\alpha,\infty} \le d(\beta,\delta)\Big(\mathscr{M}_1\left(\|X\|_\infty\right)+\mathscr{M}_2\left(\|X\|_\infty\right)\Big)$$

and

$$\begin{aligned}&\mathbf{E}\left\|(\Gamma_1+\Gamma_2)X(t_2)-(\Gamma_1+\Gamma_2)X(t_1)\right\|_\alpha^p\\ &\quad\le s(\alpha,\beta,\delta)\Big(\mathscr{M}_1\left(\|X\|_\infty\right)+\mathscr{M}_2\left(\|X\|_\infty\right)\Big)|t_2-t_1|^\gamma\end{aligned}$$

for all $X \in BC(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha))$, $t_1, t_2 \in \mathbb{R}$ with $t_1 \ne t_2$, where $d(\beta,\delta)$ and $s(\alpha,\beta,\delta)$ are positive constants. Consequently, $X \in BC(\mathbb{R}, L^p(\Omega, \mathbb{H}))$ and $\|X\|_\infty < R$ yield

$$(\Gamma_1+\Gamma_2)X \in BC^{\gamma}(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha))$$

and

$$\left\|(\Gamma_1+\Gamma_2)X\right\|_{\alpha,\infty}^p < R_1,$$

where $R_1 = c(\alpha,\beta,\delta)\Big(\mathscr{M}_1(R)+\mathscr{M}_2(R)\Big)$. Since $\mathscr{M}(R)/R \to 0$ as $R \to \infty$, and since $\mathbf{E}\|X\|^p \le c\mathbf{E}\|X\|_\alpha^p$ for all $X \in L^p(\Omega, \mathbb{H}_\alpha)$, it follows that there exists an $r > 0$ such that for all $R \ge r$, the following hold:

$$\left(\Gamma_1+\Gamma_2\right)\Big(B_{AP(\mathbb{R},L^p(\Omega,\mathbb{H}))}(0,R)\Big) \subset B_{BC^{\gamma}(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha))} \cap B_{AP(\mathbb{R},L^p(\Omega,\mathbb{H}))}(0,R). \quad (6.9)$$

In view of the above, it follows that $\left(\Gamma_1+\Gamma_2\right) : D \to D$ is continuous and compact, where D is the ball in $AP(\mathbb{R}, L^p(\Omega, \mathbb{H}))$ of radius R with $R \ge r$. Using the Schauder fixed point it follows that $\left(\Gamma_1+\Gamma_2\right)$ has a fixed point, which is obviously a p-th mean almost periodic mild solution to Eq. (5.3).

6.3.2 Existence of S^p Almost Periodic Mild Solutions

In this subsection, we introduce and develop another notion of almost periodicity known as the concept of Stepanov almost periodicity. This notion is weaker than p-th almost periodicity. Basic results on Stepanov almost periodic processes will be, subsequently, utilized to study the existence and uniqueness of Stepanov almost periodic solutions to the nonautonomous differential equations, (5.3), where $(A(t))_{t\in\mathbb{R}}$

is a family of closed linear operators on $L^p(\Omega;\mathbb{H})$ satisfying Acquistapace–Terreni conditions, and the forcing terms F,G are Stepanov almost periodic.

6.3.2.1 S^p Almost Periodic Processes

Definition 6.3. The Bochner transform $X^b(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$, of a stochastic process $X:\mathbb{R}\to L^q(\Omega;\mathscr{B})$ is defined by

$$X^b(t,s) := X(t+s).$$

Remark 6.2. A stochastic process $Z(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$, is the Bochner transform of a certain stochastic process $X(t)$,

$$Z(t,s) = X^b(t,s),$$

if and only if

$$Z(t+\tau, s-\tau) = Z(s,t)$$

for all $t\in\mathbb{R}$, $s\in[0,1]$, and $\tau\in[s-1,s]$.

Definition 6.4. Let $p,q\geq 1$. The space $BS^p(L^q(\Omega;\mathscr{B}))$ of all Stepanov bounded stochastic processes consists of all stochastic processes X on $\mathbb{R}$ with values in $L^q(\Omega;\mathscr{B})$ such that $X^b\in L^\infty\Big(\mathbb{R};L^p((0,1),L^q(\Omega;\mathscr{B}))\Big)$. This is a Banach space with the norm

$$\|X\|_{S^p} = \|X^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\mathbf{E}\big\|X(\tau)\big\|^p\,d\tau\right)^{1/p}.$$

Definition 6.5. Let $p,q\geq 1$. A stochastic process $X\in BS^p(L^q(\Omega;\mathscr{B}))$ is called Stepanov almost periodic (or S^p almost periodic) if $X^b\in AP\big(\mathbb{R};L^p((0,1),L^q(\Omega;\mathscr{B}))\big)$, that is, for each $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}\mathbf{E}\big\|X(s+\tau)-X(s)\big\|^p\,ds<\varepsilon.$$

The collection of such functions will be denoted by $S^pAP(\mathbb{R};L^q(\Omega;\mathscr{B}))$.

Throughout this section, we suppose $p=q$.
The proof of the next theorem is straightforward and hence omitted.

Theorem 6.5. *If $X:\mathbb{R}\to L^p(\Omega;\mathscr{B})$ is a p-th mean almost periodic stochastic process, then X is S^p almost periodic, that is, $AP(\mathbb{R};L^p(\Omega;\mathscr{B}))\subset S^pAP(\mathbb{R};L^p(\Omega;\mathscr{B}))$.*

Lemma 6.13. *Let $(X_n(t))_{n\in\mathbb{N}}$ be a sequence of S^p almost periodic stochastic processes such that*

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \mathbf{E}\big\|X_n(s) - X(s)\big\|^p ds \to 0, \quad as \;\; n \to \infty.$$

Then $X \in S^pAP(\mathbb{R}; L^p(\Omega; \mathscr{B}))$.

Proof. For each $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\int_t^{t+1} \mathbf{E}\|X_n(s) - X(s)\|^p \, ds \le \frac{\varepsilon}{3^p}, \;\; \forall t \in \mathbb{R}, \;\; n \ge N(\varepsilon).$$

From the S^p almost periodicity of $X_N(t)$, there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the following property:

$$\int_t^{t+1} \mathbf{E}\big\|X_N(s+\tau) - X_N(s)\big\|^p \, ds < \frac{\varepsilon}{3^p}, \;\; \forall t \in \mathbb{R}.$$

Now

$$\begin{aligned}\mathbf{E}\big\|X(t+\tau) - X(t)\big\|^p \le\; & 3^{p-1}\mathbf{E}\big\|X(t+\tau) - X_N(t+\tau)\big\|^p + 3^{p-1}\mathbf{E}\big\|X_N(t+\tau) - X_N(t)\big\|^p \\ & + 3^{p-1}\mathbf{E}\big\|X_N(t) - X(t)\big\|^p\end{aligned}$$

and hence

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \mathbf{E}\big\|X(s+\tau) - X(s)\big\|^p ds < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which completes the proof.

Similarly,

Lemma 6.14. *Let* $(X_n(t))_{n\in\mathbb{N}}$ *be a sequence of p-th mean almost periodic stochastic processes such that*

$$\sup_{s\in\mathbb{R}} \mathbf{E}\big\|X_n(s) - X(s)\big\|^p \to 0, \quad as \;\; n \to \infty.$$

Then $X \in AP(\mathbb{R}; L^p(\Omega; \mathscr{B}))$.

Using the inclusion $S^pAP(\mathbb{R}; L^p(\Omega; \mathscr{B})) \subset BS^p(\mathbb{R}; L^p(\Omega; \mathscr{B}))$ and the fact that $(BS^p(\mathbb{R}; L^p(\Omega; \mathscr{B})), \big\| \cdot \big\|_{S^p})$ is a Banach space, one can easily see that the next theorem is a straightforward consequence of Lemma 6.13.

Theorem 6.6. *The space* $S^pAP(\mathbb{R}; L^p(\Omega; \mathscr{B}))$ *equipped with the norm*

$$\big\|X\big\|_{S^p} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \mathbf{E}\big\|X(s)\big\|^p \, ds \right)^{1/p}$$

is a Banach space.

Let $(\mathscr{B}_1, \|\cdot\|_{\mathscr{B}_1})$ and $(\mathscr{B}_2, \|\cdot\|_{\mathscr{B}_2})$ be Banach spaces and let $L^p(\Omega; \mathscr{B}_1)$ and $L^p(\Omega; \mathscr{B}_2)$ be their corresponding L^p spaces, respectively.

Definition 6.6. A function $F:\mathbb{R}\times L^p(\Omega;\mathscr{B}_1)\to L^p(\Omega;\mathscr{B}_2)$, $(t,Y)\mapsto F(t,Y)$ is said to be S^p almost periodic in $t\in\mathbb{R}$ uniformly in $Y\in\mathbb{K}$ where $\mathbb{K}\subset L^p(\Omega;\mathscr{B}_1)$ is a compact if for any $\varepsilon>0$, there exists $l(\varepsilon,\mathbb{K})>0$ such that any interval of length $l(\varepsilon,\mathbb{K})$ contains at least a number τ for which

$$\sup_{t\in\mathbf{R}}\int_t^{t+1}\mathbf{E}\big\|F(s+\tau,Y)-F(s,Y)\big\|_{\mathscr{B}_2}^p\,ds<\varepsilon$$

for each stochastic process $Y:\mathbb{R}\to\mathbb{K}$.

Theorem 6.7. *Let $F:\mathbb{R}\times L^p(\Omega;\mathscr{B}_1)\to L^p(\Omega;\mathscr{B}_2)$, $(t,Y)\mapsto F(t,Y)$ be an S^p almost periodic process in $t\in\mathbb{R}$ uniformly in $Y\in\mathbb{K}$, where $\mathbb{K}\subset L^p(\Omega;\mathscr{B}_1)$ is compact. Suppose that F is Lipschitz in the following sense:*

$$\mathbf{E}\big\|F(t,Y)-F(t,Z)\big\|_{\mathscr{B}_2}^p\le M\,\mathbf{E}\big\|Y-Z\big\|_{\mathscr{B}_1}^p$$

for all $Y,Z\in L^p(\Omega;\mathscr{B}_1)$ and for each $t\in\mathbb{R}$, where $M>0$. Then for any S^p almost periodic process $\Phi:\mathbb{R}\to L^p(\Omega;\mathscr{B}_1)$, the stochastic process $t\mapsto F(t,\Phi(t))$ is S^p almost periodic.

Proof. The proof is left as an exercise.

Theorem 6.8. *Let $F:\mathbb{R}\times L^p(\Omega;\mathscr{B}_1)\to L^p(\Omega;\mathscr{B}_2)$, $(t,Y)\mapsto F(t,Y)$ be an S^p almost periodic process in $t\in\mathbb{R}$ uniformly in $Y\in K$, where $K\subset L^p(\Omega;\mathscr{B}_1)$ is any compact subset. Suppose that $F(t,\cdot)$ is uniformly continuous on bounded subsets $K'\subset L^p(\Omega;\mathscr{B}_1)$ in the following sense: for all $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that $X,Y\in K'$ and $\mathbf{E}\big\|X-Y\big\|_1^p<\delta_\varepsilon$, then*

$$\mathbf{E}\big\|F(t,Y)-F(t,Z)\big\|_2^p<\varepsilon,\ \ \forall t\in\mathbb{R}.$$

Then for any S^p almost periodic process $\Phi:\mathbb{R}\to L^p(\Omega;\mathscr{B}_1)$, the stochastic process $t\mapsto F(t,\Phi(t))$ is S^p almost periodic.

Proof. Since $\Phi:\mathbb{R}\to L^p(\Omega;\mathscr{B}_1)$ is an S^p almost periodic process, for all $\varepsilon>0$ there exists $l_\varepsilon>0$ such that every interval of length $l_\varepsilon>0$ contains a τ with the property that

$$\int_t^{t+1}\mathbf{E}\big\|\Phi(s+\tau)-\Phi(s)\big\|_1^p\,ds<\varepsilon,\ \ \forall t\in\mathbb{R}. \tag{6.10}$$

In addition, $\Phi:\mathbb{R}\to L^p(\Omega;\mathscr{B}_1)$ is bounded, that is, $\sup_{t\in R}\mathbf{E}\big\|\Phi(t)\big\|_1^p<\infty$. Let $K''\subset L^p(\Omega;\mathscr{B}_1)$ be a bounded subset such that $\Phi(t)\in K''$ for all $t\in\mathbb{R}$.

Now

$$\begin{aligned}
&\int_t^{t+1} \mathbf{E}\big\|F(s+\tau,\Phi(s+\tau)) - F(s,\Phi(s))\big\|_2^p\, ds \\
&\quad \leq 2^{p-1}\int_t^{t+1} \mathbf{E}\big\|F(s+\tau,\Phi(s+\tau)) - F(s+\tau,\Phi(s))\big\|_2^p\, ds \\
&\quad +2^{p-1}\int_t^{t+1} \mathbf{E}\big\|F(s+\tau,\Phi(s)) - F(s,\Phi(s))\big\|_2^p\, ds.
\end{aligned}$$

Taking into account Eq. (6.10) (take $\delta_\varepsilon = \varepsilon$) and using the uniform continuity of F on bounded subsets of $L^p(\Omega;\mathscr{B}_1)$ it follows that

$$\sup_{t\in\mathbb{R}}\int_t^{t+1} \mathbf{E}\big\|F(s+\tau,\Phi(s+\tau)) - F(s+\tau,\Phi(s))\big\|_2^p\, ds < \frac{\varepsilon}{2^p}. \tag{6.11}$$

Similarly, using the S^p almost periodicity of F it follows that

$$\sup_{t\in\mathbb{R}}\int_t^{t+1} \mathbf{E}\big\|F(s+\tau,\Phi(s)) - F(s,\Phi(s))\big\|_2^p\, ds < \frac{\varepsilon}{2^p}. \tag{6.12}$$

Combining Eqs. (6.11) and (6.12) one obtains that

$$\sup_{t\in\mathbb{R}}\int_t^{t+1} \mathbf{E}\big\|F(s+\tau,\Phi(s+\tau)) - F(s,\Phi(s))\big\|_2^p\, ds < \varepsilon,$$

and hence the stochastic process $t\mapsto F(t,\Phi(t))$ is S^p almost periodic.

6.3.2.2 Existence of S^p Almost Periodic Mild Solutions

To study S^p almost periodic solutions to Eq. (5.3), we first study the existence of S^p almost periodic solutions to the stochastic nonautonomous differential equations

$$dX(t) = A(t)X(t)dt + f(t)dt + g(t)d\mathbb{W}(t), \quad t\in\mathbb{R}, \tag{6.13}$$

where $A(t)$ for $t\in\mathbb{R}$ is a family of closed linear operators where the family of linear operator $A(t): D(A(t))\subset L^p(\Omega;\mathbb{H})\to L^p(\Omega;\mathbb{H})$ satisfies the above-mentioned assumptions and the forcing terms $f\in S^pAP(\mathbb{R},L^p(\Omega;\mathbb{H}))\cap C(\mathbb{R},L^p(\Omega;\mathbb{H}))$ and $g\in S^pAP(\mathbb{R},L^p(\Omega;\mathbb{L}_2^0))\cap C(\mathbb{R},L^p(\Omega;\mathbb{L}_2^0))$. In addition to $(5H)_3$ and $(6H)_6$, we require the following assumptions.

$(6H)_{12}$ $R(\zeta,A(\cdot))\in S^pAP(L^p(\Omega;\mathbb{H}))$.

$(6H)_{13}$ The function $F:\mathbb{R}\times L^p(\Omega,\mathbb{H})\to L^p(\Omega,\mathbb{H})$ is S^p almost periodic in the first variable uniformly in the second variable. Furthermore, $X\to F(t,X)$ is uniformly continuous on any bounded subset $\mathscr{O}$ of $L^p(\Omega,\mathbb{H})$ for each $t\in\mathbb{R}$. Finally,

$$\sup_{t\in\mathbb{R}}\mathbf{E}\big\|F(t,X)\big\|^p \leq \mathscr{M}_1\big(\|X\|_\infty\big)$$

where $\mathscr{M}_1:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r\to\infty} \frac{\mathscr{M}_1(r)}{r} = 0\,.$$

$(6H)_{14}$ The function $G : \mathbb{R} \times L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{L}_2^0)$ is S^p almost periodic in the first variable uniformly in the second variable. Furthermore, $X \to G(t,X)$ is uniformly continuous on any bounded subset $\mathscr{O}'$ of $L^p(\Omega, \mathbb{H})$ for each $t \in \mathbb{R}$. Finally,

$$\sup_{t\in\mathbb{R}} \mathbf{E}\big\|G(t,X)\big\|^p \le \mathscr{M}_2\big(\big\|X\big\|_\infty\big)$$

where $\mathscr{M}_2 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r\to\infty} \frac{\mathscr{M}_2(r)}{r} = 0\,.$$

Theorem 6.9. *Assume that* (2.38), (2.39)*, and* $(5H)_3$ *hold. Then* (6.13) *has a unique bounded solution* $X \in S^pAP(\mathbb{R}, L^p(\Omega;\mathbb{H}))$.

We need the following lemmas. For the proofs of Lemmas 6.15 and 6.16, see the proof of Theorem 6.10.

Lemma 6.15. *Under the assumptions of Theorem 6.9, the integral defined by*

$$X_n(t) = \int_{n-1}^{n} U(t,t-\xi) f(t-\xi)\, d\xi$$

belongs to $S^pAP(\mathbb{R}, L^p(\Omega;\mathbb{H}))$ *for each for* $n = 1, 2, \ldots$*.*

Lemma 6.16. *Under the assumptions of Theorem 6.9, the integral defined by*

$$Y_n(t) = \int_{n-1}^{n} U(t,t-\xi) g(t-\xi)\, d\mathbb{W}(\xi)$$

belongs to $S^pAP(\mathbb{R}, L^p(\Omega;\mathbb{L}_2^0))$ *for each for* $n = 1, 2, \ldots$*.*

Proof. (Theorem 6.9) By assumption there exist some constants $M, \delta > 0$ such that

$$\|U(t,s)\| \le Me^{-\delta(t-s)} \quad \text{for every} \quad t \ge s.$$

Let us first prove uniqueness. Assume that $X : \mathbb{R} \to L^p(\Omega;\mathbb{H})$ is a bounded stochastic process that satisfies the homogeneous equation

$$dX(t) = A(t)X(t)dt, \quad t \in \mathbb{R}. \tag{6.14}$$

Then $X(t) = U(t,s)X(s)$ for any $t \ge s$. Hence $\big\|X(t)\big\| \le MDe^{-\delta(t-s)}$ with $\big\|X(s)\big\| \le D$ for $s \in \mathbb{R}$ almost surely. Take a sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to -\infty$ as $n \to \infty$. For any $t \in \mathbb{R}$ fixed, one can find a subsequence $(s_{n_k})_{k\in\mathbb{N}} \subset (s_n)_{n\in\mathbb{N}}$ such that $s_{n_k} < t$ for all $k = 1, 2, \ldots$. By letting $k \to \infty$, we get $X(t) = 0$ almost surely.

Now, if $X_1, X_2 : \mathbb{R} \to L^p(\Omega;\mathbb{H})$ are bounded solutions to Eq. (6.13), then $X = X_1 - X_2$ is a bounded solution to Eq. (6.14). In view of the above, $X = X_1 - X_2 = 0$ almost surely, that is, $X_1 = X_2$ almost surely.

Now let us investigate the existence. Consider for each $n = 1,2,...$, the integrals

$$X_n(t) = \int_{n-1}^{n} U(t,t-\xi)f(t-\xi)d\xi$$

and

$$Y_n(t) = \int_{n-1}^{n} U(t,t-\xi)g(t-\xi)d\mathbb{W}(\xi).$$

First, we know by Lemma 6.15 that the sequence X_n belongs to $S^pAP(\mathbb{R},L^p(\Omega;\mathbb{H}))$. Moreover, note that

$$\begin{aligned}\int_t^{t+1} \mathbf{E}\|X_n(s)\|^p\,ds &\leq \int_t^{t+1} \mathbf{E}\left\|\int_{n-1}^{n} U(s,s-\xi)\,f(s-\xi)\,d\xi\right\|^p ds\\ &\leq M^p \int_{n-1}^{n} e^{-p\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|f(s-\xi)\|^p\,ds\right\}d\xi\\ &\leq M^p\|f\|_{S^p}^p\left\{\int_{n-1}^{n} e^{-p\delta\xi}\,d\xi\right\}\\ &\leq \frac{M^p}{p\delta}\,\|f\|_{S^p}^p\,e^{-p\delta n}(e^{p\delta}+1)\,.\end{aligned}$$

Since the series

$$\frac{M^p}{p\delta}\,(e^{p\delta}+1)\sum_{n=2}^{\infty} e^{-p\delta n}$$

is convergent, it follows from the Weierstrass test that the sequence of partial sums defined by

$$L_n(t) := \sum_{k=1}^{n} X_k(t)$$

converges in sense of the norm $\|\cdot\|_{S^p}$ uniformly on $\mathbb{R}$.

Now let

$$l(t) := \sum_{n=1}^{\infty} X_n(t)$$

for each $t \in \mathbb{R}$.

Observe that

$$l(t) = \int_{-\infty}^{t} U(t,\xi)f(\xi)d\xi,\ \ t\in\mathbb{R},$$

and hence $l \in C(\mathbb{R};L^p(\Omega,\mathbb{H}))$.

Similarly, the sequence Y_n belongs to $S^pAP(\mathbb{R},L^p(\Omega;\mathbb{L}_2^0))$. Moreover, note that

$$\begin{aligned}\int_t^{t+1}\mathbf{E}\|Y_n(s)\|^p\,ds &\le C_p\int_t^{t+1}\mathbf{E}\Big[\int_{n-1}^n \|U(s,s-\xi)\|^2\,\|g(s-\xi)\|^2\,d\xi\Big]^{p/2}ds\\ &\le C_p M^p\int_{n-1}^n e^{-p\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|g(s-\xi)\|^p\,ds\right\}d\xi\\ &\le C_p\frac{M^p}{p\delta}\,\|g\|_{S^p}^p\,e^{-p\delta n}(e^{p\delta}+1)\,.\end{aligned}$$

Proceeding as before we can show easily that the sequence of partial sums defined by

$$M_n(t) := \sum_{k=1}^n Y_k(t)$$

converges in sense of the norm $\|\cdot\|_{S^p}$ uniformly on $\mathbb{R}$.

Now let

$$m(t) := \sum_{n=1}^\infty Y_n(t)$$

for each $t\in\mathbb{R}$.

Observe that

$$m(t) = \int_{-\infty}^t U(t,\xi)g(\xi)d\mathbb{W}(\xi),\ \ t\in\mathbb{R},$$

and hence $m\in C(\mathbb{R},L^p(\Omega;\mathbb{L}_2^0))$.

Setting

$$X(t) = \int_{-\infty}^t U(t,\xi)f(\xi)\,d\xi + \int_{-\infty}^t U(t,\xi)g(\xi)\,d\mathbb{W}(\xi),$$

one can easily see that X is a bounded solution to Eq. (6.13). Moreover,

$$\int_t^{t+1}\mathbf{E}\|X(s)-(L_n(s)+M_n(s))\|^p ds \to 0 \ \text{ as } \ n\to\infty$$

uniformly in $t\in\mathbb{R}$, and hence using Lemma 6.13, it follows that X is a S^p almost periodic solution. In view of the above, it follows that X is the only bounded S^p almost periodic solution to Eq. (6.13).

Definition 6.7. An $\mathscr{F}_t$-progressively process $\{X(t)\}_{t\in\mathbb{R}}$ is called a mild solution of (5.3) on $\mathbb{R}$ if

$$\begin{aligned}X(t) = U(t,s)X(s) &+ \int_s^t U(t,\sigma)F(\sigma,X(\sigma))\,d\sigma\\ &+ \int_s^t U(t,\sigma)G(\sigma,X(\sigma))\,d\mathbb{W}(\sigma)\end{aligned} \tag{6.15}$$

for all $t\ge s$ for each $s\in\mathbb{R}$.

Now, define the nonlinear integral operators Γ on $S^pAP(\mathbb{R}, L^p(\Omega, \mathbb{H}))$as follows:

$$\Gamma_1 X(t) = \Gamma_1 X(t) + \Gamma_2 X(t)$$

where

$$(\Gamma_1 X)(t) := \int_{-\infty}^{t} U(t,s)F(s,X(s))\,ds$$

and

$$(\Gamma_2 X)(t) := \int_{-\infty}^{t} U(t,s)G(s,X(s))\,d\mathbb{W}(s)\,.$$

Throughout this section we assume that $\alpha \in \left(0, \frac{1}{2} - \frac{1}{p}\right)$ if $p > 2$ and $\alpha \in \left(0, \frac{1}{2}\right)$ if $p = 2$. Moreover, we suppose that

$$2\beta > \alpha + 1\,.$$

Lemma 6.17. *Under assumptions* $(5H)_3$, $(6H)_6$, $(6H)_9$, $(6H)_{13}$, *and* $(6H)_{14}$, *the mappings* $\Gamma_i (i = 1,2) : BC(\mathbb{R}, L^p(\Omega, \mathbb{H})) \to BC(\mathbb{R}, L^p(\Omega, \mathbb{H}_\alpha))$ *are well defined and continuous.*

Proof. The proof follows along the same lines as that of Lemma 6.7 and hence is omitted.

Lemma 6.18. *Under assumptions* $(5H)_3$, $(6H)_6$, $(6H)_9$, $(6H)_{13}$, $(6H)_{14}$, *the integral operator* $\Gamma_i (i = 1,2)$ *maps* $S^pAP\big(\mathbb{R}, L^p(\Omega, \mathbb{H})\big)$ *into itself.*

Proof. Consider for each $n = 1, 2, \ldots$, the integral

$$R_n(t) = \int_{n-1}^{n} U(t,t-\xi)f(t-\xi)\,d\xi + \int_{n-1}^{n} U(t,t-\xi)g(t-\xi)\,d\mathbb{W}(\xi)\,,$$

where $f(\sigma) = F(\sigma, X(\sigma))$ and $g(\sigma) = G(\sigma, X(\sigma))$.

Set

$$X_n(t) = \int_{n-1}^{n} U(t,t-\xi)f(t-\xi)\,d\xi$$

and

$$Y_n(t) = \int_{n-1}^{n} U(t,t-\xi)g(t-\xi)\,d\mathbb{W}(\xi)\,.$$

Let us first show that $X_n(\cdot)$ is S^p almost periodic whenever X is. Indeed, assuming that X is S^p almost periodic and using $(6H)_{13}$, Theorem 6.8, and Lemma 5.1, given $\varepsilon > 0$, one can find $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least τ with the property that

$$\big\| U(t+\tau, s+\tau) - U(t,s) \big\| \le \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \ge \varepsilon$, and

$$\int_t^{t+1} \mathbf{E}\|f(s+\tau)-f(s)\|^p\,ds < \eta(\varepsilon)$$

for each $t \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

For the S^p almost periodicity of $X_n(\cdot)$, we need to consider two cases.

Case 1: $n \geq 2$

$$\begin{aligned}
&\int_t^{t+1} \mathbf{E}\|X_n(s+\tau)-X_n(s)\|^p\,ds \\
&= \int_t^{t+1} \mathbf{E}\Big\|\int_{n-1}^{n} U(s+\tau,s+\tau-\xi)\,f(s+\tau-\xi)\,d\xi \\
&- \int_{n-1}^{n} U(s,s-\xi)\,f(s-\xi)\,d\xi\Big\|^p\,ds \\
&\leq 2^{p-1}\int_t^{t+1}\int_{n-1}^{n} \|U(s+\tau,s+\tau-\xi)\|^p\mathbf{E}\|f(s+\tau-\xi)-f(s-\xi)\|^p\,d\xi\,ds \\
&+2^{p-1}\int_t^{t+1}\int_{n-1}^{n} \|U(s+\tau,s+\tau-\xi)-U(s,s-\xi)\|^p\mathbf{E}\|f(s-\xi)\|^p\,d\xi\,ds \\
&\leq 2^{p-1}M^p\int_t^{t+1}\int_{n-1}^{n} e^{-p\delta\xi}\,\mathbf{E}\|f(s+\tau-\xi)-f(s-\xi)\|^p\,d\xi\,ds \\
&+2^{p-1}\varepsilon^p\int_t^{t+1}\int_{n-1}^{n} e^{-\frac{p}{2}\delta\xi}\mathbf{E}\|f(s-\xi)\|^p\,d\xi\,ds \\
&\leq 2^{p-1}M^p\int_{n-1}^{n} e^{-p\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|f(s+\tau-\xi)-f(s-\xi)\|^p\,ds\right\}d\xi \\
&+2^{p-1}\varepsilon^p\int_{n-1}^{n} e^{-\frac{p}{2}\delta\xi}\left\{\int_t^{t+1}\mathbf{E}\|f(s-\xi)\|^p\,ds\right\}d\xi.
\end{aligned}$$

Case 2: $n = 1$

We have

$$\begin{aligned}
&\int_t^{t+1} \mathbf{E}\|X_1(s+\tau)-X_1(s)\|^p\,ds \\
&= \int_t^{t+1} \mathbf{E}\Big\|\int_0^1 U(s+\tau,s+\tau-\xi)\,f(s+\tau-\xi)\,d\xi - \int_0^1 U(s,s-\xi)\,f(s-\xi)\,d\xi\Big\|^p\,ds \\
&\leq 3^{p-1}\int_t^{t+1}\int_0^1 \|U(s+\tau,s+\tau-\xi)\|^p\mathbf{E}\|f(s+\tau-\xi)-f(s-\xi)\|^p\,d\xi\,ds \\
&+3^{p-1}\int_t^{t+1}\int_\varepsilon^1 \|U(s+\tau,s+\tau-\xi)-U(s,s-\xi)\|^p\mathbf{E}\|f(s-\xi)\|^p\,d\xi\,ds \\
&+3^{p-1}\int_t^{t+1}\int_0^\varepsilon \|U(s+\tau,s+\tau-\xi)-U(s,s-\xi)\|^p\mathbf{E}\|f(s-\xi)\|^p\,d\xi\,ds
\end{aligned}$$

$$\le 3^{p-1} M^p \int_t^{t+1} \int_0^1 e^{-p\delta\xi}\, \mathbf{E}\big\|f(s+\tau-\xi) - f(s-\xi)\big\|^p \, d\xi \, ds$$
$$+3^{p-1} \varepsilon^p \int_t^{t+1} \int_\varepsilon^1 e^{-\frac{p}{2}\delta\xi} \mathbf{E}\big\|f(s-\xi)\big\|^p \, d\xi \, ds$$
$$+6^{p-1} M^p \int_t^{t+1} \int_0^\varepsilon e^{-p\delta\xi} \mathbf{E}\big\|f(s-\xi)\big\|^p \, d\xi \, ds$$
$$\le 3^{p-1} M^p \int_0^1 e^{-p\delta\xi} \left\{ \int_t^{t+1} \mathbf{E}\big\|f(s+\tau-\xi) - f(s-\xi)\big\|^p \, ds \right\} d\xi$$
$$+3^{p-1} \varepsilon^p \int_\varepsilon^1 e^{-\frac{p}{2}\delta\xi} \left\{ \int_t^{t+1} \mathbf{E}\big\|f(s-\xi)\big\|^p \, ds \right\} d\xi$$
$$+6^{p-1} M^p \int_0^\varepsilon e^{-p\delta\xi} \left\{ \int_t^{t+1} \mathbf{E}\big\|f(s-\xi)\big\|^p \, ds \right\} d\xi$$

which implies that $X_n(\cdot)$ is S^p almost periodic.

Similarly, assuming that X is S^p almost periodic and using $(6H)_{14}$, Theorem 6.8, and Lemma 5.1, given $\varepsilon > 0$, one can find $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least τ with the property that

$$\big\|U(t+\tau, s+\tau) - U(t,s)\big\| \le \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \ge \varepsilon$, and

$$\int_t^{t+1} \mathbf{E}\big\|g(s+\tau) - g(s)\big\|_{\mathbb{L}_2^0}^p \, ds < \eta(\varepsilon)$$

for each $t \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The next step consists in proving the S^p almost periodicity of $Y_n(\cdot)$. Here again, we need to consider two cases.

Case 1: $n \ge 2$

For $p > 2$, we have

$$\int_t^{t+1} \mathbf{E}\big\|Y_n(s+\tau) - Y_n(s)\big\|^p \, ds$$
$$= \int_t^{t+1} \mathbf{E}\bigg\| \int_{n-1}^n U(s+\tau, s+\tau-\xi)\, g(s+\tau-\xi)\, d\mathbb{W}(\xi)$$
$$- \int_{n-1}^n U(s, s-\xi)\, g(s-\xi)\, dW(\xi) \bigg\|^p \, ds$$
$$\le 2^{p-1} C_p \int_t^{t+1} \mathbf{E}\Big[\int_{n-1}^n \big\|U(s+\tau, s+\tau-\xi)\big\|^2 \big\|g(s+\tau-\xi) - g(s-\xi)\big\|_{\mathbb{L}_2^0}^2 d\xi \Big]^{p/2} ds$$
$$+2^{p-1} C_p \int_t^{t+1} \mathbf{E}\Big[\int_{n-1}^n \big\|U(s+\tau, s+\tau-\xi) - U(s, s-\xi)\big\|^2 \big\|g(s-\xi)\big\|_{\mathbb{L}_2^0}^2 d\xi \Big]^{p/2} ds$$

$$\leq 2^{p-1} M^p \int_t^{t+1} \mathbf{E}\Big[\int_{n-1}^{n} e^{-2\delta\xi} \, \|g(s+\tau-\xi) - g(s-\xi)\|^2_{\mathbb{L}_2^0} \, d\xi\Big]^{p/2} ds$$

$$+2^{p-1} \varepsilon^p \int_t^{t+1} \mathbf{E}\Big[\int_{n-1}^{n} e^{-\delta\xi} \mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0} \, d\xi\Big]^{p/2} ds$$

$$\leq 2^{p-1} M^p \int_{n-1}^{n} e^{-p\delta\xi} \left\{\int_t^{t+1} \mathbf{E}\|g(s+\tau-\xi) - g(s-\xi)\|^p_{\mathbb{L}_2^0} \, ds\right\} d\xi$$

$$+2^{p-1} \varepsilon^p \int_{n-1}^{n} e^{-\frac{p}{2}\delta\xi} \left\{\int_t^{t+1} \mathbf{E}\|g(s-\xi)\|^p_{\mathbb{L}_2^0} \, ds\right\} d\xi.$$

For $p = 2$, a simple computation using Itô isometry identity shows that

$$\int_t^{t+1} \mathbf{E}\|Y_n(s+\tau) - Y_n(s)\|^2 \, ds$$

$$\leq 2M^2 \int_{n-1}^{n} e^{-2\delta\xi} \Big\{\int_t^{t+1} \mathbf{E}\|g(s+\tau-\xi) - g(s-\xi)\|^2_{\mathbb{L}_2^0} \, ds\Big\} d\xi$$

$$+2\varepsilon^2 \int_{n-1}^{n} e^{-\delta\xi} \Big\{\int_t^{t+1} \mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\Big\}.$$

Case 2: $n = 1$

For $p > 2$, we have

$$\int_t^{t+1} \mathbf{E}\|Y_1(s+\tau) - Y_1(s)\|^p \, ds$$

$$= \int_t^{t+1} \mathbf{E}\Big\| \int_0^1 U(s+\tau, s+\tau-\xi)\, g(s+\tau-\xi)\, d\mathbb{W}(\xi)$$

$$- \int_n^{n+1} U(s, s-\xi)\, g(s-\xi)\, d\mathbb{W}(\xi)\Big\|^p ds$$

$$\leq 3^{p-1} C_p \int_t^{t+1} \mathbf{E}\Big[\int_0^1 \|U(s+\tau, s+\tau-\xi)\|^2 \mathbf{E}\|g(s+\tau-\xi) - g(s-\xi)\|^2_{\mathbb{L}_2^0} d\xi\Big]^{p/2} ds$$

$$+3^{p-1} C_p \int_t^{t+1} \mathbf{E}\Big[\int_\varepsilon^1 \|U(s+\tau, s+\tau-\xi) - U(s, s-\xi)\|^2 \mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0} d\xi\Big]^{p/2} ds$$

$$+3^{p-1} C_p \int_t^{t+1} \mathbf{E}\Big[\int_0^\varepsilon \|U(s+\tau, s+\tau-\xi) - U(s, s-\xi)\|^2 \mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0} d\xi\Big]^{p/2} ds$$

$$\leq 3^{p-1} M^p C_p \int_t^{t+1} \mathbf{E}\Big[\int_0^1 e^{-2\delta\xi} \mathbf{E}\|g(s+\tau-\xi) - g(s-\xi)\|^2_{\mathbb{L}_2^0} d\xi\Big]^{p/2} ds$$

$$+3^{p-1} \varepsilon^p C_p \int_t^{t+1} \mathbf{E}\Big[\int_\varepsilon^1 e^{-\delta\xi} \mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0} d\xi\Big]^{p/2} ds$$

$$+6^{p-1} M^p C_p \int_t^{t+1} \mathbf{E}\Big[\int_0^\varepsilon e^{-2\delta\xi} \mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0} d\xi\Big]^{p/2} ds$$

$$\leq 3^{p-1} M^p C_p \int_0^1 e^{-p\delta\xi} \left\{\int_t^{t+1} \mathbf{E}\|g(s+\tau-\xi) - g(s-\xi)\|^p_{\mathbb{L}_2^0} ds\right\} d\xi$$

$$+3^{p-1}\ \varepsilon^p C_p \int_\varepsilon^1 e^{-\frac{p}{2}\delta\xi}\left\{\int_t^{t+1} \mathbf{E}\|g(s-\xi)\|^p_{\mathbb{L}_2^0}\, ds\right\} d\xi$$
$$+6^{p-1} M^p C_p \int_0^\varepsilon e^{-p\delta\xi}\left\{\int_t^{t+1} \mathbf{E}\|g(s-\xi)\|^p_{\mathbb{L}_2^0}\, ds\right\} d\xi.$$

For $p=2$, a simple calculation shows that

$$\begin{aligned}\int_t^{t+1} \mathbf{E}\|Y_1(s+\tau)-Y_1(s)\|^2\, ds \leq 3M^2 \int_0^1 e^{-2\delta\xi}\Big\{\int_t^{t+1} \mathbf{E}\|g(s+\tau-\xi)-g(s-\xi)\|^2_{\mathbb{L}_2^0}\, ds\Big\} d\xi \\ +3\varepsilon^2 \int_\varepsilon^1 e^{-\delta\xi}\Big\{\int_t^{t+1} \mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\Big\} d\xi \\ +6M^2 \int_0^\varepsilon e^{-\delta\xi}\Big\{\int_t^{t+1} \mathbf{E}\|g(s-\xi)\|^2_{\mathbb{L}_2^0}\Big\} d\xi\end{aligned}$$

which implies that $Y_n(\cdot)$ is S^p almost periodic.

Setting

$$\Gamma X(t) := \int_{-\infty}^t U(t,\sigma)F(\sigma,X(\sigma))\, d\sigma + \int_{-\infty}^t U(t,\sigma)G(\sigma,X(\sigma))\, d\mathbb{W}(\sigma)$$

and proceeding as in the proof of Theorem 6.9, one can easily see that

$$\int_t^{t+1} \mathbf{E}\|X(s)-(X_n(s)+Y_n(s))\|^p ds \to 0 \ \text{ as } \ n\to\infty$$

uniformly in $t\in\mathbb{R}$, and hence using Lemma 6.13, it follows that ΓX is an S^p almost periodic solution.

Let $\gamma\in(0,1]$ and let $BC^\gamma\big(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha)\big) = \Big\{X\in BC\big(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha)\big) : \|X\|_{\alpha,\gamma} < \infty\Big\}$, where

$$\|X\|_{\alpha,\gamma} = \sup_{t\in\mathbb{R}} \Big[\mathbf{E}\|X(t)\|^p_\alpha\Big]^{\frac{1}{p}} + \gamma \sup_{t,\,s\in\mathbb{R},\, s\neq t} \frac{\Big[\mathbf{E}\|X(t)-X(s)\|^p_\alpha\Big]^{\frac{1}{p}}}{|t-s|^\gamma}.$$

Clearly, the space $BC^\gamma\big(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha)\big)$ equipped with the norm $\|\cdot\|_{\alpha,\gamma}$ is a Banach space, which is in fact the Banach space of all bounded continuous Hölder functions from $\mathbb{R}$ to $L^p(\Omega,\mathbb{H}_\alpha)$ whose Hölder exponent is γ.

Lemma 6.19. *Under assumptions* $(5H)_3$, $(6H)_6$, $(6H)_9$, $(6H)_{13}$, *and* $(6H)_{14}$, *the mapping* $\Gamma_i (i=1,2)$ *defined previously map bounded sets of* $BC\big(\mathbb{R},L^p(\Omega,\mathbb{H})\big)$ *into bounded sets of* $BC^\gamma(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha))$ *for some* $0<\gamma<1$.

Proof. The proof is almost identical to that of Lemmas 6.9 and 6.10 and may be omitted.

Lemma 6.20. *The integral operators $\Gamma_i (i=1,2)$ map bounded sets of $AP(\Omega, L^p(\Omega,\mathbb{H}))$ into bounded sets of $BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)) \cap S^pAP(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ for $0<\gamma<\alpha$.*

Proof. The proof follows along the same lines as that of Lemma 6.9 and hence is omitted.

Similarly, the next lemma is a consequence of [80, Proposition 3.3]. Note in this context that $\mathbb{X} = L^p(\Omega,\mathbb{H})$ and $\mathbb{Y} = L^p(\Omega,\mathbb{H}_\alpha)$.

Lemma 6.21. *For $0<\gamma<\alpha$, $BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ is compactly contained in $BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$, that is, the canonical injection*

$$id : BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)) \to BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$$

is compact, which yields

$$id : BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha)) \cap S^pAP(\mathbb{R}, L^p(\Omega,\mathbb{H})) \to AP(\mathbb{R}, L^p(\Omega,\mathbb{H}))$$

is compact, too.

Theorem 6.10. *Suppose assumptions $(5H)_3$, $(6H)_6$, $(6H)_9$, $(6H)_{13}$, and $(6H)_{14}$ hold, then the nonautonomous differential equation* (5.3) *has at least one S^p almost periodic solution.*

Proof. Let us recall that in view of Lemmas 6.7 and 6.8, we have

$$\left\|(\Gamma_1+\Gamma_2)X\right\|_{\alpha,\infty} \le d(\beta,\delta)\Big(\mathscr{M}_1\left(\|X\|_\infty\right)+\mathscr{M}_2\left(\|X\|_\infty\right)\Big)$$

and

$$\begin{aligned}&\mathbf{E}\left\|(\Gamma_1+\Gamma_2)X(t_2)-(\Gamma_1+\Gamma_2)X(t_1)\right\|_\alpha^p\\ &\quad\le s(\alpha,\beta,\delta)\Big(\mathscr{M}_1\left(\|X\|_\infty\right)\Big)+\mathscr{M}_2\left(\|X\|_\infty\right)\Big)|t_2-t_1|^\gamma\end{aligned}$$

for all $X \in BC(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$, $t_1, t_2 \in \mathbb{R}$ with $t_1 \neq t_2$, where $d(\beta,\delta)$ and $s(\alpha,\beta,\delta)$ are positive constants. Consequently, $X \in BC(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ and $\|X\|_\infty < R$ yield $(\Gamma_1+\Gamma_2)X \in BC^\gamma(\mathbb{R}, L^p(\Omega,\mathbb{H}_\alpha))$ and $\left\|(\Gamma_1+\Gamma_2)X\right\|_{\alpha,\infty}^p < R_1$ where $R_1 = c(\alpha,\beta,\delta)\Big(\mathscr{M}_1(R)+\mathscr{M}_2(R)\Big)$. Since $\mathscr{M}(R)/R \to 0$ as $R\to\infty$, and since $\mathbf{E}\|X\|^p \le c\mathbf{E}\|X\|_\alpha^p$ for all $X \in L^p(\Omega,\mathbb{H}_\alpha)$, it follows that there exists an $r>0$ such that for all $R \ge r$, the following holds:

$$(\Gamma_1+\Gamma_2)\Big(B_{S^pAP(\mathbb{R},L^p(\Omega,\mathbb{H}))}(0,R)\Big) \subset B_{BC^\gamma(\mathbb{R},L^p(\Omega,\mathbb{H}_\alpha))} \cap B_{S^pAP(\mathbb{R},L^p(\Omega,\mathbb{H}))}(0,R)\,.$$

In view of the above, it follows that $\big(\Gamma_1+\Gamma_2\big) : D \to D$ is continuous and compact, where D is the ball in $S^pAP(\mathbb{R}, L^p(\Omega,\mathbb{H}))$ of radius R with $R \ge r$. Using the Schauder fixed point it follows that $\big(\Gamma_1+\Gamma_2\big)$ has a fixed point, which is obviously a p-th mean almost periodic mild solution to Eq. (5.3).

The next result is weaker than Theorem 6.10 although we require that G be bounded in some sense.

Theorem 6.11. *Under assumptions* $(5H)_3$, $(6H)_6$, $(6H)_9$, $(6H)_{13}$, $(6H)_{14}$, *if we assume that there exists* $L > 0$ *such that* $\mathbf{E}\|G(t,Y)\|^p_{\mathbb{L}_2^0} \leq L$ *for all* $t \in \mathbb{R}$ *and* $Y \in L^p(\Omega;\mathbb{H})$, *then Eq.* (5.3) *has a unique p-th mean almost periodic mild solution, which can be explicitly expressed as follows:*

$$X(t) = \int_{-\infty}^{t} U(t,\sigma)F(\sigma,X(\sigma))\,d\sigma + \int_{-\infty}^{t} U(t,\sigma)G(\sigma,X(\sigma))\,d\mathbb{W}(\sigma) \text{ for each } t \in \mathbb{R}$$

whenever K *and* K' *are small enough.*

Proof. We use the same notations as in the proof of Theorem 6.10. Let us first show that $X_n(\cdot)$ is p-th mean almost periodic upon the S^p almost periodicity of $f = F(\cdot,X(\cdot))$. Indeed, assuming that X is S^p almost periodic and using $(6H)_{13}$, Theorem 6.8, and Lemma 5.1, given $\varepsilon > 0$, one can find $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least τ with the property that

$$\left\|U(t+\tau,s+\tau) - U(t,s)\right\| \leq \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \geq \varepsilon$, and

$$\int_{t}^{t+1} \mathbf{E}\left\|f(s+\tau) - f(s)\right\|^p ds < \eta(\varepsilon)$$

for each $t \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The next step consists in proving the p-th mean almost periodicity of $X_n(\cdot)$. Here again, we need to consider two cases.

Case 1: $n \geq 2$

$$\begin{aligned}
&\mathbf{E}\left\|X_n(t+\tau) - X_n(t)\right\|^p \\
&= \mathbf{E}\left\| \int_{n-1}^{n} U(t+\tau,t+\tau-\xi)\, f(t+\tau-\xi)\, d\xi \right. \\
&\left. - \int_{n-1}^{n} U(t,t-\xi)\, f(s-\xi)\, d\xi \right\|^p \\
&\leq 2^{p-1} \int_{n-1}^{n} \left\|U(t+\tau,t+\tau-\xi)\right\|^p \mathbf{E}\left\|f(t+\tau-\xi) - f(t-\xi)\right\|^p d\xi \\
&+ 2^{p-1} \int_{n-1}^{n} \left\|U(t+\tau,t+\tau-\xi) - U(t,t-\xi)\right\|^p \mathbf{E}\left\|f(t-\xi)\right\|^p d\xi \\
&\leq 2^{p-1} M^p \int_{n-1}^{n} e^{-p\delta\xi}\, \mathbf{E}\left\|f(t+\tau-\xi) - f(t-\xi)\right\|^p d\xi \\
&+ 2^{p-1} \varepsilon^p \int_{n-1}^{n} e^{-\frac{p}{2}\delta\xi} \mathbf{E}\left\|f(t-\xi)\right\|^p d\xi
\end{aligned}$$

$$\leq 2^{p-1} M^p \int_{t-n+1}^{t-n} \mathbf{E}\big\|f(r+\tau)-f(r)\big\|^p\,dr + 2^{p-1}\,\varepsilon^p \int_{t-n+1}^{t-n} \mathbf{E}\big\|f(r)\big\|^p\,dr$$

Case 2: $n=1$

$$\begin{aligned}
&\mathbf{E}\big\|X_1(t+\tau)-X_1(t)\big\|^p\\
&=\mathbf{E}\left\|\int_0^1 U(t+\tau,t+\tau-\xi)\,f(t+\tau-\xi)\,d\xi-\int_0^1 U(t,t-\xi)\,f(t-\xi)\,d\xi\right\|^p\\
&\leq 3^{p-1}\,\mathbf{E}\left[\int_0^1 \big\|U(t+\tau,t+\tau-\xi)\big\|\,\big\|f(t+\tau-\xi)-f(t-\xi)\big\|\,d\xi\right]^p\\
&+3^{p-1}\,\mathbf{E}\left[\int_\varepsilon^1 \big\|U(t+\tau,t+\tau-\xi)-U(t,t-\xi)\big\|\,\big\|f(t-\xi)\big\|\,d\xi\right]^p\\
&+3^{p-1}\,\mathbf{E}\left[\int_0^\varepsilon \big\|U(t+\tau,t+\tau-\xi)-U(t,t-\xi)\big\|\,\big\|f(t-\xi)\big\|\,d\xi\right]^p\\
&\leq 3^{p-1}\,M^p\,\mathbf{E}\left[\int_0^1 e^{-\delta\xi}\,\big\|f(t+\tau-\xi)-f(t-\xi)\big\|\,d\xi\right]^p\\
&+3^{p-1}\,\varepsilon^p\,\mathbf{E}\left[\int_\varepsilon^1 e^{-\frac{\delta}{2}\xi}\,\big\|f(t-\xi)\big\|\,d\xi\right]^p+6^{p-1}M^p\,\mathbf{E}\left[\int_0^\varepsilon e^{-\delta\xi}\,\big\|f(t-\xi)\big\|\,d\xi\right]^p.
\end{aligned}$$

Now, using Hölder's inequality, we have

$$\begin{aligned}
&\leq 3^{p-1}\,M^p\left(\int_0^1 e^{-p\delta\xi}\mathbf{E}\big\|f(t+\tau-\xi)-f(t-\xi)\big\|^p\,d\xi\right)\\
&+3^{p-1}\,\varepsilon^p\left(\int_\varepsilon^1 e^{-p\frac{\delta}{2}\xi}\mathbf{E}\big\|f(t-\xi)\big\|^p\,d\xi\right)\\
&+6^{p-1}\,M^p\left(\int_0^\varepsilon e^{-p\delta\xi}\mathbf{E}\big\|f(t-\xi)\big\|^p\,d\xi\right)\\
&\leq 3^{p-1}\,M^p\int_{t-1}^t \mathbf{E}\big\|f(r+\tau)-f(r)\big\|^p\,dr\\
&+3^{p-1}\,\varepsilon^p\int_{t-1}^{t-\varepsilon}\mathbf{E}\big\|f(r)\big\|^p\,dr+6^{p-1}\,M^p\,\varepsilon\int_{t-\varepsilon}^t \mathbf{E}\big\|f(r)\big\|^p\,dr,
\end{aligned}$$

which implies that $X_n(\cdot)$ is p-th mean almost periodic.

Similarly, using $(6H)_{14}$, Theorem 6.8, and Lemma 5.1, given $\varepsilon>0$, one can find $l(\varepsilon)>0$ such that any interval of length $l(\varepsilon)$ contains at least τ with the property that

$$\big\|U(t+\tau,s+\tau)-U(t,s)\big\|\leq \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t-s\geq\varepsilon$, and

$$\int_t^{t+1}\mathbf{E}\big\|g(s+\tau)-g(s)\big\|_{\mathbb{L}_2^0}^p\,ds<\eta$$

for each $t \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, there exists a positive constant $L > 0$ such that

$$\sup_{\sigma \in \mathbb{R}} \mathbf{E}\|g(\sigma)\|^p_{\mathbb{L}_2^0} \le L.$$

The next step consists in proving the p-th mean almost periodicity of $Y_n(\cdot)$.

Case 1: $n \ge 2$

For $p > 2$, we have

$$\begin{aligned}
&\mathbf{E}\|Y_n(t+\tau) - Y_n(t)\|^p \\
&= \mathbf{E}\left\|\int_{n-1}^{n} U(t+\tau, t+\tau-\xi) g(s+\tau-\xi)\, dW(\xi) - \int_{n-1}^{n} U(t, t-\xi)\, g(t-\xi)\, d\mathbb{W}(\xi)\right\|^p \\
&\le 2^{p-1} C_p \mathbf{E}\Big[\int_{n-1}^{n} \|U(t+\tau, t+\tau-\xi)\|^2 \|g(t+\tau-\xi) - g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi\Big]^{p/2} \\
&+ 2^{p-1} C_p \mathbf{E}\Big[\int_{n-1}^{n} \|U(t+\tau, t+\tau-\xi) - U(t, t-\xi)\|^2 \|g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi\Big]^{p/2} \\
&\le 2^{p-1} C_p M^p \mathbf{E}\Big[\int_{n-1}^{n} e^{-2\delta\xi} \|g(t+\tau-\xi) - g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi\Big]^{p/2} \\
&+ 2^{p-1} C_p \varepsilon^p \mathbf{E}\Big[\int_{n-1}^{n} e^{-\delta\xi} \mathbf{E}\|g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi\Big]^{p/2} \\
&\le 2^{p-1} M^p \int_{t-n}^{t-n+1} \mathbf{E}\|g(r+\tau) - g(r)\|^p_{\mathbb{L}_2^0}\, dr + 2^{p-1} C_p \varepsilon^p \int_{t-n}^{t-n+1} \mathbf{E}\|g(r)\|^p_{\mathbb{L}_2^0}\, dr.
\end{aligned}$$

For $p = 2$, a simple calculation using Itô isometry identity shows

$$\mathbf{E}\|Y_n(t+\tau) - Y_n(t)\|^2 \le 2M^2 \int_{t-n}^{t-n+1} \mathbf{E}\|g(r+\tau) - g(r)\|^2_{\mathbb{L}_2^0}\, dr + 2\varepsilon^2 \int_{t-n}^{t-n+1} \mathbf{E}\|g(r)\|^2_{\mathbb{L}_2^0}\, dr.$$

Case 2: $n = 1$

For $p > 2$, we have

$$\begin{aligned}
&\mathbf{E}\|Y_1(t+\tau) - Y_1(t)\|^p \\
&= \mathbf{E}\left\|\int_0^1 U(t+\tau, t+\tau-\xi) g(s+\tau-\xi)\, d\mathbb{W}(\xi) - \int_0^1 U(t, t-\xi) g(t-\xi)\, d\mathbb{W}(\xi)\right\|^p \\
&\le 3^{p-1} C_p \mathbf{E}\Big[\int_0^1 \|U(t+\tau, t+\tau-\xi)\|^2 \|g(t+\tau-\xi) - g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi\Big]^{p/2} \\
&+ 3^{p-1} C_p \int_t^{t+1} \left(\int_\varepsilon^1 + \int_0^\varepsilon\right) \|U(t+\tau, t+\tau-\xi) - U(t, t-\xi)\|^2 \|g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi\Big]^{p/2} \\
&\le 3^{p-1} C_p M^p \mathbf{E}\Big[\int_0^1 e^{-2\delta\xi} \|g(t+\tau-\xi) - g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi\Big]^{p/2} \\
&+ 3^{p-1} C_p \varepsilon^p \mathbf{E}\Big[\int_\varepsilon^1 e^{-\delta\xi} \|g(t-\xi)\|^2_{\mathbb{L}_2^0}\, d\xi\Big]^{p/2}
\end{aligned}$$

$$
\begin{aligned}
&+6^{p-1} C_p M^p \mathbf{E}\Big[\int_0^{\varepsilon} e^{-2\delta\xi} \big\|g(t-\xi)\big\|^2_{\mathbb{L}_2^0} d\xi\Big]^{p/2} \\
&\leq 3^{p-1} C_p M^p \int_0^1 \mathbf{E}\big\|g(t+\tau-\xi)-g(t-\xi)\big\|^p_{\mathbb{L}_2^0} d\xi \\
&+3^{p-1} C_p \varepsilon^p \int_{\varepsilon}^1 \mathbf{E}\big\|g(t-\xi)\big\|^p_{\mathbb{L}_2^0} d\xi + 6^{p-1} C_p M^p \int_0^{\varepsilon} \mathbf{E}\big\|g(t-\xi)\big\|^p_{\mathbb{L}_2^0} d\xi \\
&\leq 3^{p-1} C_p M^p \int_{t-1}^t \mathbf{E}\big\|g(r+\tau)-g(r)\big\|^p_{\mathbb{L}_2^0} dr \\
&+3^{p-1} C_p \varepsilon^p \int_{t-1}^t \mathbf{E}\big\|g(r)\big\|^p_{\mathbb{L}_2^0} dr + 6^{p-1} C_p M^p \int_0^{\varepsilon} \mathbf{E}\big\|g(t-\xi)\big\|^p_{\mathbb{L}_2^0} d\xi \\
&\leq 3^{p-1} C_p M^p \int_{t-1}^t \mathbf{E}\big\|g(r+\tau)-g(r)\big\|^p_{\mathbb{L}_2^0} dr \\
&+3^{p-1} C_p \varepsilon^p \int_{t-1}^t \mathbf{E}\big\|g(r)\big\|^p_{\mathbb{L}_2^0} dr + 6^{p-1} \varepsilon C_p M^p L.
\end{aligned}
$$

For $p=2$, we have

$$
\begin{aligned}
&\mathbf{E}\big\|Y_1(t+\tau)-Y_1(t)\big\|^2 \\
&\quad 3M^2 \int_{t-1}^t \mathbf{E}\big\|g(r+\tau)-g(r)\big\|^2_{\mathbb{L}_2^0} dr \\
&\quad +3\varepsilon^2 \int_{t-1}^t \mathbf{E}\big\|g(r)\big\|^2_{\mathbb{L}_2^0} + 6M^2 \int_0^{\varepsilon} \mathbf{E}\big\|g(t-\xi)\big\|^2_{\mathbb{L}_2^0} d\xi,
\end{aligned}
$$

which implies that $Y_n(\cdot)$ is p-th mean almost periodic. Moreover, setting

$$
\Gamma X(t) = \int_{-\infty}^t U(t,\sigma)F(\sigma,X(\sigma))\,d\sigma + \int_{-\infty}^t U(t,\sigma)G(\sigma,X(\sigma))\,d\mathbb{W}(\sigma)
$$

for each $t \in \mathbb{R}$ and proceeding as in the proofs of Theorems 6.4 and 6.10, one can easily see that

$$
\sup_{s\in\mathbb{R}} \mathbf{E}\big\|X(s)-(X_n(s)+Y_n(s))\,\big\|^p \to 0 \ \text{ as } \ n \to \infty
$$

and it follows that ΓX is a p-th mean almost periodic solution to Eq. (5.3).

In view of the above, the nonlinear operator Γ maps $AP(\mathbb{R}; L^p(\Omega;\mathscr{B}))$ into itself. Consequently, using the Schauder fixed-point principle it follows that Γ has a unique fixed point $\{X_1(t),\ t \in \mathbb{R}\}$, which in fact is the only p-th mean almost periodic solution to Eq. (5.3).

6.3.3 Example

Here we reconsider Example 5.2.3. For the sake of clarity, we reproduce it here. Indeed, let $\mathscr{O} \subset \mathbb{R}^n$ be a bounded subset whose boundary $\partial\mathscr{O}$ is both of class C^2 and locally on one side of $\mathscr{O}$. Of interest is the following stochastic parabolic partial differential equation:

$$d_t X(t,x) = A(t,x)X(t,x)d_t + F(t,X(t,x))d_t + G(t,X(t,x))\, d\mathbb{W}(t), \quad (6.16)$$

$$\sum_{i,j=1}^{n} n_i(x)a_{ij}(t,x)d_i X(t,x) = 0, \;\; t \in \mathbb{R}, \;\; x \in \partial\mathscr{O}, \quad (6.17)$$

where $d_t = \dfrac{d}{dt}$, $d_i = \dfrac{d}{dx_i}$, $n(x) = (n_1(x), n_2(x), ..., n_n(x))$ is the outer unit normal vector, the family of operators $A(t,x)$ are formally given by

$$A(t,x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j}\right) + c(t,x), \;\; t \in \mathbb{R}, \;\; x \in \mathscr{O},$$

$\mathbb{W}$ is a real-valued Brownian motion, and a_{ij}, c $(i,j = 1,2,...,n)$ satisfy the following conditions:

We require the following assumptions:

$(6H)_{15}$ The coefficients $(a_{ij})_{i,j=1,...,n}$ are symmetric, that is, $a_{ij} = a_{ji}$ for all $i,j = 1,...,n$. Moreover,

$$a_{ij} \in C_b^{\mu}(\mathbb{R}, L^2(\Omega, C(\overline{\mathscr{O}}))) \cap BC(\mathbb{R}, L^2(\Omega, C^1(\overline{\mathscr{O}}))) \cap S^2AP(\mathbb{R}; L^2(\Omega, L^2(\mathscr{O})))$$

for all $i,j = 1,...n$, and

$$c \in C_b^{\mu}(\mathbb{R}, L^2(\Omega, L^2(\mathscr{O}))) \cap BC(\mathbb{R}, L^2(\Omega, C(\overline{\mathscr{O}}))) \cap S^2AP(\mathbb{R}; L^2(\Omega, L^1(\mathscr{O})))$$

for some $\mu \in (1/2, 1]$.

$(6H)_{16}$ There exists $\delta_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\eta_i\,\eta_j \geq \delta_0|\eta|^2,$$

for all $(t,x) \in \mathbb{R} \times \overline{\mathscr{O}}$ and $\eta \in \mathbb{R}^n$.

Now let $\mathbb{H} = L^2(\mathscr{O})$ and let $H^2(\mathscr{O})$ be the Sobolev space of order 2 on $\mathscr{O}$. For each $t \in \mathbb{R}$, define an operator $A(t)$ on $L^2(\Omega; \mathbb{H})$ by

$$\mathscr{D}(A(t)) = \left\{X \in L^2(\Omega, H^2(\mathscr{O})) : \sum_{i,j=1}^{n} n_i(\cdot)a_{ij}(t,\cdot)d_i X(t,\cdot) = 0 \;\text{ on }\; \partial\mathscr{O}\right\} \;\text{ and}$$

$$A(t)X = A(t,x)X(x), \text{ for all } X \in \mathscr{D}(A(t)) .$$

The next corollary is a consequence of Theorem 6.9.

Corollary 6.1. *Under previous assumptions, the system* (6.16)–(6.17) *has a unique mild solution, which obviously is S^2 almost periodic, whenever M is small enough.*

Similarly,

Corollary 6.2. *Under previous assumptions, if we suppose that there exists $L > 0$ such that $\mathbf{E}\|G(t,Y)\|^2_{\mathbb{L}_2^0} \le L$ for all $t \in \mathbb{R}$ and $Y \in L^2(\Omega, L^2(\mathscr{O}))$, the system* (6.16)–(6.17) *has a unique square mean almost periodic solution, whenever M is small enough.*

6.4 Bibliographical Notes

All the main results presented in this chapter are based on some recent work by the authors, see, e.g., [22, 21] and Diagana [55].

Chapter 7
Existence Results For Some Second-Order Stochastic Differential Equations

This chapter is devoted to the study of the solutions of (non)autonomous second-order stochastic differential equations. The existence of solutions to second-order stochastic differential equations is important due to possible applications. In this chapter, we adopt the same notations as in Chapter 6. In addition, if $\mathfrak{L}$ is a family of 2×2–operator matrices defined on $\mathbb{H}\times\mathbb{H}$, we then define the corresponding family L of operator matrices on $L^2(\Omega,\mathbb{H}\times\mathbb{H})$ as follows:

$$Z \in D(L) \ \text{ and } \ LZ = W$$

if and only if

$$Z,W \in L^2(\Omega,\mathbb{H}\times\mathbb{H}) \ \text{ and } \ \mathfrak{L}Z(\omega) = W(\omega) \ \text{ for all } \ \omega\in\Omega.$$

7.1 Square-Mean Almost Periodic Solutions to Autonomous Second-Order SDEs

7.1.1 Introduction

The principal motivation of the present work comes from two main sources from the deterministic setting. The first one is a paper by Mawhin [138], in which the dissipativeness and the existence of bounded solutions on the whole real number line to the second-order differential equations given by

$$u''(t) + cu'(t) + Au + g(t,u) = 0, \quad t \in \mathbb{R}, \tag{7.1}$$

where $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a self-adjoint operator on a Hilbert space $\mathbb{H}$, which is semi-positive definite and has a compact resolvent, $c > 0$, and $g : \mathbb{R}\times\mathbb{H}\to\mathbb{H}$ is bounded, sufficiently regular, and satisfies some semi-coercivity condition, was established. The abstract results in [138] were subsequently utilized to study the ex-

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istence of bounded solutions to the so-called nonlinear telegraph equation subject to some Neumann boundary conditions. Unfortunately, the main result of this section does not apply to the telegraph equation as the linear operator presented in [138], which involves Neumann boundary conditions, lacks exponential dichotomy.

The second source is a paper by Leiva [117], in which the existence of (exponentially stable) bounded solutions and almost periodic solutions to the second-order systems of differential equations given by

$$u''(t)+cu'(t)+dAu+kH(u)=P(t),\;\; u\in\mathbb{R}^n,\;\; t\;\;\in\mathbb{R}, \tag{7.2}$$

where A is an $n\times n$–matrix whose eigenvalues are positive, c,d,k are positive constants, $H:\mathbb{R}^n\to\mathbb{R}^n$ is a locally Lipschitz function, $P:\mathbb{R}\to\mathbb{R}^n$ is a bounded continuous function, was established.

In this section, using slightly different techniques as in [14, 118], we study and obtain some reasonable sufficient conditions, which do guarantee the existence of square-mean almost periodic solutions to the classes of autonomous second-order stochastic differential equations

$$\begin{aligned} dX'(\omega,t)+a\,dX(\omega,t) &= \Big[-b\mathscr{A}X(\omega,t)+f(t,X(\omega,t))\Big]\,dt \\ &\quad +g(t,X(\omega,t))\,d\mathbb{W}(\omega,t), \end{aligned} \tag{7.3}$$

for all $\omega\in\Omega$ and $t\in\mathbb{R}$, where $\mathscr{A}:D(\mathscr{A})\subset\mathbb{H}\to\mathbb{H}$ is a self-adjoint linear operator whose spectrum consists of isolated eigenvalues

$$0<\lambda_1<\lambda_2<...<\lambda_n\to\infty$$

with each eigenvalue having a finite multiplicity γ_j equal to the multiplicity of the corresponding eigenspace, the functions $a,b>0$ are constants, and the functions $f,g:\mathbb{R}\times L^2(\Omega,\mathbb{H})\to L^2(\Omega,\mathbb{H})$ are jointly continuous functions satisfying some additional conditions and $\mathbb{W}$ is a one-dimensional Brownian motion. For that, the main idea consists of rewriting Eq. (7.3) as an autonomous first-order differential equation on $\mathbb{H}\times\mathbb{H}$ involving the family of 2×2–operator matrices $\mathfrak{L}$. Indeed, setting

$$Z:=\begin{pmatrix} X \\ dX(t)\end{pmatrix},$$

then Eq. (7.3) can be rewritten in the Hilbert space $\mathbb{H}\times\mathbb{H}$ in the following form:

$$dZ(\omega,t)=\Bigg[\mathfrak{L}Z(\omega,t)+F(t,Z(\omega,t))\Bigg]\,dt+G(t,Z(\omega,t))d\mathbb{W}(\omega,t),\; t\in\mathbb{R}, \tag{7.4}$$

where $\mathfrak{L}$ is the family of 2×2-operator matrices defined on $\mathscr{H}=\mathbb{H}\times\mathbb{H}$ by

$$\mathfrak{L} = \begin{pmatrix} 0 & I_{\mathbb{H}} \\ & \\ & \\ -b\mathscr{A} & -aI_{\mathbb{H}} \end{pmatrix} \tag{7.5}$$

whose domain $D(\mathfrak{L})$ is given by $D(\mathfrak{L}) = D(\mathscr{A}) \times \mathbb{H}$. Moreover, the semilinear terms F, G appearing in Eq. (7.4) are defined on $\mathbb{R} \times \mathscr{H}_\alpha$ for some $\alpha \in (0,1)$ by

$$F(t,Z) = \begin{pmatrix} 0 \\ f(t,X) \end{pmatrix}, \qquad G(t,Z) = \begin{pmatrix} 0 \\ g(t,X) \end{pmatrix},$$

where $\mathscr{H}_\alpha = \tilde{\mathscr{H}_\alpha} \times \mathbb{H}$ with $\tilde{\mathscr{H}_\alpha}$ the real interpolation space between $\mathbb{H}$ and $D(\mathscr{A})$ given by

$$\tilde{\mathscr{H}_\alpha} := \Big(\mathbb{H}, D(\mathscr{A})\Big)_{\alpha,\infty}.$$

Under some reasonable assumptions, it will be shown that the linear operator matrix $\mathfrak{L}$ is sectorial. Moreover, it will be shown that its corresponding analytic semigroup $T(t)$ is exponentially stable under those assumptions.

7.1.2 Preliminaries

In this section, $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{H} \to \mathbb{H}$ stands for a self-adjoint linear operator whose spectrum consists of isolated eigenvalues $0 < \lambda_1 < \lambda_2 < ... < \lambda_n \to \infty$ with each eigenvalue having a finite multiplicity γ_j equal to the multiplicity of the corresponding eigenspace.

Let $\{e_j^k\}$ be a (complete) orthonormal sequence of eigenvectors associated with the eigenvalues $\{\lambda_j\}_{j\geq 1}$.

Clearly, for each $u \in D(\mathscr{A}) := \left\{ x \in \mathbb{H} : \quad \sum_{j=1}^{\infty} \lambda_j^2 \left\| E_j x \right\|^2 < \infty \right\}$,

$$\mathscr{A}x = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle x, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j x$$

where $E_j x = \sum_{k=1}^{\gamma_j} \langle x, e_j^k \rangle e_j^k$.

Note that $\{E_j\}_{j\geq 1}$ is a sequence of orthogonal projections on $\mathbb{H}$. Moreover, each $x \in \mathbb{H}$ can written as follows:

$$x = \sum_{j=1}^{\infty} E_j x.$$

It should also be mentioned that the operator $-\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$, which is explicitly expressed in terms of those orthogonal projections E_j by, for all $x \in \mathbb{H}$,

$$T(t)x = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j x.$$

In addition, the fractional powers $\mathscr{A}^r$ $(r \geq 0)$ of $\mathscr{A}$ exist and are given by

$$D(\mathscr{A}^r) = \left\{ x \in \mathbb{H} : \sum_{j=1}^{\infty} \lambda_j^{2r} \left\| E_j x \right\|^2 < \infty \right\}$$

and

$$\mathscr{A}^r x = \sum_{j=1}^{\infty} \lambda_j^{2r} E_j x, \ \ \forall x \in D(\mathscr{A}^r).$$

7.1.3 The Abstract Setting

To analyze Eq. (7.4), our strategy consists in studying the existence of square-mean almost periodic solutions to the corresponding class of stochastic differential equations of the form

$$dZ(t) = \Big[LZ(t) + F(t, Z(t)) \Big] dt + G(t, Z(t)) d\mathbb{W}(t) \tag{7.6}$$

for all $t \in \mathbb{R}$, where $L : D(L) \subset L^2(\Omega, \mathscr{H}_\alpha) \to L^2(\Omega, \mathscr{H})$ is a sectorial linear operator whose corresponding analytic semigroup is hyperbolic, that is, $\sigma(L) \cap i\mathbb{R} = \emptyset$, F, G as before, and $\mathbb{W}$ is a one-dimensional Brownian motion.

We adopt the following assumptions.

$(7H)_1$ The operator $\mathscr{A}$ is sectorial and generates a hyperbolic (analytic) semigroup $(T(t))_{t\geq 0}$.

$(7H)_2$ Let $\alpha \in (0, \frac{1}{2})$. Then $\mathbb{H}_\alpha = D((-\mathscr{A})^\alpha)$, or $\mathbb{H}_\alpha = D_{\mathscr{A}}(\alpha, p)$, $1 \leq p \leq \infty$, or $\mathbb{H}_\alpha = D_{\mathscr{A}}(\alpha)$, or $\mathbb{H}_\alpha = [\mathbb{H}, D(\mathscr{A})]_\alpha$.

$(7H)_3$ Let $\alpha \in (0, \frac{1}{2})$ and $\alpha < \beta < 1$. Let f, $g : \mathbb{R} \times L^2(\Omega; \mathbb{H}_\alpha) \to L^2(\Omega; \mathbb{H})$ be square-mean almost periodic. Moreover, the functions f and g are uniformly Lipschitz with respect to the second argument in the following sense: there exist positive constants K_f and K_g such that

$$\mathbf{E}\big\| f(t,X) - f(t,Y) \big\|^2 \leq K_f \, \mathbf{E}\big\| X - Y \big\|^2, \ \text{ and}$$

$$\mathbf{E}\big\| g(t,X) - g(t,Y) \big\|^2 \leq K_g \, \mathbf{E}\big\| X - Y \big\|^2$$

for all stochastic processes $X, Y \in L^2(\Omega; \mathbb{H}_\alpha)$ and $t \in \mathbb{R}$.

Theorem 7.1. *Under assumptions* $(7H)_1$, $(7H)_2$, *and* $(7H)_3$, *the evolution equation (7.6) has a unique square-mean almost periodic mild solution whenever* $\Theta < 1$, *where* Θ *is the appropriate constant appearing in Theorem 6.14.*

Proof. The proof follows along the same lines as the proof of Theorem 6.1 and hence omitted.

7.1.4 Existence of Square-Mean Almost Periodic Solutions

We have previously seen that each $X \in L^2(\Omega, \mathbb{H})$ can be written in terms of the sequence of orthogonal projections E_n as follows:

$$X = \sum_{n=1}^{\infty} \sum_{k=1}^{\gamma_n} \langle X, e_n^k \rangle e_n^k = \sum_{n=1}^{\infty} E_n X.$$

Moreover, for each $X \in D(A)$,

$$AX = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle X, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j X.$$

Therefore, for all $Z := \begin{pmatrix} X \\ Y \end{pmatrix} \in D(L) = D(A) \times L^2(\Omega, \mathbb{H})$, we obtain the following:

$$\begin{aligned}
LZ &= \begin{pmatrix} 0 & I_{L^2(\Omega,\mathbb{H})} \\ -bA & -aI_{L^2(\Omega,\mathbb{H})} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\
&= \begin{pmatrix} Y \\ -bAX - aY \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} E_n Y \\ -b \sum_{n=1}^{\infty} \lambda_n E_n X - a \sum_{n=1}^{\infty} E_n Y \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ -b\lambda_n & -a \end{pmatrix} \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\
&= \sum_{n=1}^{\infty} A_n P_n Z,
\end{aligned}$$

where

$$P_n := \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix}, \; n \geq 1,$$

and

$$A_n := \begin{pmatrix} 0 & 1 \\ -b\lambda_n & -a \end{pmatrix}, \; n \geq 1. \tag{7.7}$$

Now, the characteristic equation for A_n is given by

$$\lambda^2 + a\lambda + \lambda_n b = 0. \tag{7.8}$$

In this section we suppose $a, b > 0$ and

$$a^2 < 4\lambda_1 b. \tag{7.9}$$

From Eq. (7.9) it easily follows that the discriminant of Eq. (7.8) defined by $\Delta_n = a^2 - 4\lambda_n b < 0$ for all $t \in \mathbb{R}$, $n \geq 1$, and hence all roots of Eq. (7.8) are nonzero (with nonzero real and imaginary parts) complex roots given by

$$\lambda_1^n = \frac{-a + i\sqrt{-\Delta_n}}{2} \text{ and } \lambda_2^n = \overline{\lambda_1^n} = \frac{-a - i\sqrt{-\Delta_n}}{2},$$

that is,

$$\sigma(A_n) = \left\{\lambda_1^n, \lambda_2^n\right\}.$$

Setting $\tilde{\theta} := \tan^{-1}\left(\dfrac{a}{2\sqrt{4\lambda_1 b - a^2}}\right)$, then

$$0 < \tilde{\theta} < \frac{\pi}{2}.$$

Define

$$S_\theta = \left\{z \in \mathbb{C} \setminus \{0\} : \left|\arg z\right| \leq \theta\right\},$$

where $\theta = \dfrac{\pi}{2} + \tilde{\theta} \in \left(\dfrac{\pi}{2}, \pi\right)$.

On the other hand, one can show without difficulty that $A_n = K_n^{-1} J_n K_n$, where J_n, K_n and K_n^{-1} are respectively given by

$$J_n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}, \quad K_n = \begin{pmatrix} 1 & 1 \\ \lambda_1^n & \lambda_2^n \end{pmatrix},$$

and

$$K_n^{-1} = \frac{1}{\lambda_1^n - \lambda_2^n} \begin{pmatrix} -\lambda_2^n & 1 \\ \lambda_1^n & -1 \end{pmatrix}.$$

For $\lambda \in S_\theta$ and $Z \in L^2(\Omega, \mathscr{H})$, one has

$$
\begin{aligned}
R(\lambda,L)Z &= \sum_{n=1}^{\infty}(\lambda-A_n)^{-1}P_nZ \\
&= \sum_{n=1}^{\infty}K_nP_n(\lambda-J_nP_n)^{-1}K_n^{-1}P_nZ.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbf{E}\left\|R(\lambda,L)Z\right\|^2 &\leq \sum_{n=1}^{\infty}\left\|K_nP_n(\lambda-J_nP_n)^{-1}K_n^{-1}P_n\right\|^2_{B(\mathscr{H})}\mathbf{E}\left\|P_nZ\right\|^2 \\
&\leq \sum_{n=1}^{\infty}\left\|K_nP_n\right\|^2\left\|(\lambda-J_nP_n)^{-1}\right\|^2\left\|K_n^{-1}P_n\right\|^2\mathbf{E}\left\|P_nZ\right\|^2.
\end{aligned}
$$

Clearly, for $Z := \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in L^2(\Omega,\mathscr{H})$, there exists $C_1 > 0$ such that

$$
\mathbf{E}\left\|K_nP_nZ\right\|^2 \leq C_1\lambda_n^1\mathbf{E}\left\|Z\right\|^2 \quad \text{for all } \ n \geq 1.
$$

Similarly, for $Z := \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in L^2(\Omega,\mathscr{H})$, one can show that there is $C_2 > 0$ such that

$$
\mathbf{E}\left\|K_n^{-1}P_nZ\right\|^2 \leq \frac{C_2}{\lambda_n}\mathbf{E}\left\|Z\right\|^2 \quad \text{for all } \ n \geq 1.
$$

Now, for $Z \in L^2(\Omega,\mathscr{H})$, we have

$$
\begin{aligned}
\mathbf{E}\left\|(\lambda-J_nP_n)^{-1}Z\right\|^2 &= \mathbf{E}\left\|\begin{pmatrix} \frac{1}{\lambda-\lambda_n^1} & 0 \\ 0 & \frac{1}{\lambda-\lambda_n^2} \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right\|^2 \\
&\leq \frac{1}{\left|\lambda-\lambda_n^1\right|^2}\mathbf{E}\left\|Z_1\right\|^2 + \frac{1}{\left|\lambda-\lambda_n^2\right|^2}\mathbf{E}\left\|Z_2\right\|^2.
\end{aligned}
$$

Let $\lambda_0 > 0$. Define the function

$$
\eta(\lambda) := \frac{1+\left|\lambda\right|}{\left|\lambda-\lambda_n^2\right|}.
$$

It is clear that the function η is continuous and bounded on the closed set

$$
\Sigma := \left\{\lambda \in \mathbb{C} : \ \left|\lambda\right| \leq \lambda_0, \ \left|\arg\lambda\right| \leq \theta\right\}.
$$

On the other hand, it is clear that the function η is bounded for $|\lambda| > \lambda_0$. Thus the function η is bounded on S_θ. If we take

$$N = \sup\left\{ \frac{1+|\lambda|}{|\lambda - \lambda_n^j|} \;:\; \lambda \in S_\theta, n \geq 1;\; j = 1,2, \right\},$$

then

$$\mathbf{E}\left\|(\lambda - J_n P_n)^{-1} Z\right\|^2 \leq \frac{N}{1+|\lambda|}\mathbf{E}\left\|Z\right\|^2, \quad \lambda \in S_\theta.$$

Consequently,

$$\left\|R(\lambda, L)\right\| \leq \frac{K}{1+|\lambda|}$$

for all $\lambda \in S_\theta$.

Note that the operator L is invertible with

$$L^{-1} = \begin{pmatrix} -ab^{-1}A^{-1} & -b^{-1}A^{-1} \\ & \\ I_{L^2(\Omega,\mathbb{H})} & 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

First of all, note that L generates an analytic semigroup $(e^{\tau L})_{\tau \geq 0}$ on $\mathscr{H}$ given by

$$e^{\tau L} Z = \sum_{n=0}^{\infty} K_n^{-1} P_n e^{\tau J_n} P_n K_n P_n Z,\; Z \in L^2(\Omega, \mathscr{H}).$$

On the other hand, we have

$$\mathbf{E}\left\|e^{\tau A} Z\right\|^2 = \sum_{n=0}^{\infty} \left\|K_n^{-1} P_n\right\|^2_{B(\mathscr{H})} \left\|e^{\tau J_n} P_n\right\|^2_{B(\mathscr{H})} \left\|K_n P_n\right\|^2_{B(\mathscr{H})} \mathbf{E}\left\|P_n Z\right\|^2,$$

with for each $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$

$$\begin{aligned} \mathbf{E}\left\|e^{\tau J_n} P_n Z\right\|^2 &= \mathbf{E}\left\| \begin{pmatrix} e^{\lambda_n^1 \tau} E_n & 0 \\ & \\ 0 & e^{\lambda_n^2 \tau} E_n \end{pmatrix} \begin{pmatrix} Z_1 \\ \\ Z_2 \end{pmatrix} \right\|^2 \\ &\leq \mathbf{E}\left\|e^{\lambda_n^1 \tau} E_n Z_1\right\|^2 + \mathbf{E}\left\|e^{\lambda_n^2 \tau} E_n Z_2\right\|^2 \\ &\leq e^{-2\delta\tau}\mathbf{E}\left\|Z\right\|^2, \end{aligned}$$

where $\delta = \dfrac{a}{4}$.

Therefore,

$$\left\| e^{\tau L} \right\| \leq C e^{-\delta \tau}, \quad \tau \geq 0. \tag{7.10}$$

It is now clear that if $\Theta < 1$ is small enough, then the second-order differential equation (7.4) has a unique solution

$$\binom{X}{Y} \in L^2(\Omega, \mathscr{H}_\alpha) = L^2(\Omega, \tilde{\mathscr{H}}_\alpha \times \mathbb{H}),$$

which in addition is square-mean almost periodic. Therefore, Eq. (7.3) has a unique bounded solution $X(t) \in L^2(\Omega, \tilde{\mathscr{H}}_\alpha),\ t \in \mathbb{R}$, which in addition is square-mean almost periodic.

7.2 Square-Mean Almost Periodic Solutions to Nonautonomous Second-Order SDEs

In this section we study and obtain under some reasonable assumptions, the existence of square-mean almost periodic solutions to some classes of nonautonomous second-order stochastic differential equations on a Hilbert space. Unlike in Section 7.1, here we make extensive use of the Schauder fixed-point theorem and the ideas and techniques developed in Goldstein–N'Guérékata [80] and Diagana [55].

7.2.1 Introduction

Of concern is the study of the classes of nonautonomous second-order stochastic differential equations

$$\begin{aligned} dX'(\omega,t) + a(t)\,dX(\omega,t) = {} & \Big[-b(t)\,\mathscr{A}X(\omega,t) + f_1(t, X(\omega,t)) \Big]\, dt \\ & + f_2(t, X(\omega,t))\, d\mathbb{W}(\omega,t), \end{aligned} \tag{7.11}$$

for all $\omega \in \Omega$ and $t \in \mathbb{R}$, where $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{H} \to \mathbb{H}$ is a self-adjoint linear operator whose spectrum consists of isolated eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \to \infty$ with each eigenvalue having a finite multiplicity γ_j equal to the multiplicity of the corresponding eigenspace, the functions $a, b : \mathbb{R} \to (0, \infty)$ are almost periodic functions, and the functions $f_i (i = 1,2) : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$ are jointly continuous functions satisfying some additional conditions and $\mathbb{W}$ is a one-dimensional Brownian motion.

For that, the main idea consists of rewriting Eq. (7.11) as a nonautonomous first-order differential equation on $\mathbb{H} \times \mathbb{H}$ involving the family of $2{\times}2$–operator matrices

$\mathfrak{L}(t)$. Indeed, setting $Z := \begin{pmatrix} X \\ dX(t) \end{pmatrix}$, Eq. (7.11) can be rewritten in the Hilbert space $\mathbb{H} \times \mathbb{H}$ in the following form:

$$dZ(\omega,t) = \Big[\mathfrak{L}(t)Z(\omega,t) + F_1(t,Z(\omega,t))\Big]\,dt + F_2(t,Z(\omega,t))d\mathbb{W}(\omega,t),\ t \in \mathbb{R}, \tag{7.12}$$

where $\mathfrak{L}(t)$ is the family of 2×2–operator matrices defined on $\mathscr{H} = \mathbb{H} \times \mathbb{H}$ by

$$\mathfrak{L}(t) = \begin{pmatrix} 0 & I_{\mathbb{H}} \\ & \\ & \\ -b(t)\mathscr{A} & -a(t)I_{\mathbb{H}} \end{pmatrix} \tag{7.13}$$

whose domain $D = D(\mathfrak{L}(t))$ is constant in $t \in \mathbb{R}$ and is given by $D(\mathfrak{L}(t)) = D(\mathscr{A}) \times \mathbb{H}$. Moreover, the semilinear term $F_i(i = 1,2)$ appearing in Eq. (7.12) is defined on $\mathbb{R} \times \mathscr{H}$ for some $\alpha \in (0,1)$ by

$$F_i(t,Z) = \begin{pmatrix} 0 \\ f_i(t,X) \end{pmatrix}.$$

To study the existence of square-mean solutions of Eq. (7.12), in addition to $(6H)_6$ we adopt the following assumptions.

$(7H)_4$ The injection $\mathbb{H}_\alpha \hookrightarrow \mathbb{H}$ is compact.

$(7H)_5$ Let $f_i(i = 1,2) : \mathbb{R} \times L^2(\Omega;\mathbb{H}) \to L^2(\Omega;\mathbb{H})$ be square-mean almost periodic. Furthermore, $X \mapsto f_i(t,X)$ is uniformly continuous on any bounded subset K of $L^2(\Omega;\mathbb{H})$ for each $t \in \mathbb{R}$. Finally,

$$\sup_{t\in\mathbb{R}} \mathbf{E}\big\|f_i(t,X)\big\|^2 \le \mathscr{M}_i\big(\|X\|_\infty\big)$$

where $\mathscr{M}_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r\to\infty} \frac{\mathscr{M}_i(r)}{r} = 0\,.$$

Under the above assumptions, it will be shown that the linear operator matrices $\mathfrak{L}(t)$ satisfy the well-known Acquistapace–Terreni conditions, which does guarantee the existence of an evolution family $\mathfrak{U}(t,s)$ associated with it. Moreover, it will be shown that $\mathfrak{U}(t,s)$ is exponentially stable under these assumptions.

Throughout this section we assume that $0 < \alpha < \frac{1}{2} < \beta < 1$ with $2\beta > \alpha + 1$.

7.2.2 Square-Mean Almost Periodic Solutions

To analyze Eq. (7.12), our strategy consists in studying the existence of square-mean almost periodic solutions to the corresponding class of stochastic differential equations of the form

$$dZ(t) = \Big[L(t)Z(t) + F_1(t, Z(t))\Big]dt + F_2(t, Z(t))d\mathbb{W}(t) \tag{7.14}$$

for all $t \in \mathbb{R}$, where the operators $L(t) : D(L(t)) \subset L^2(\Omega, \mathscr{H}) \to L^2(\Omega, \mathscr{H})$ satisfy Acquistapace–Terreni conditions, $F_i (i = 1, 2)$ as before, and $\mathbb{W}$ is a one-dimensional Brownian motion.

Note that each $Z \in L^2(\Omega, \mathbb{H})$ can be written in terms of the sequence of orthogonal projections E_n as follows:

$$X = \sum_{n=1}^{\infty} \sum_{k=1}^{\gamma_n} \langle X, e_n^k \rangle e_n^k = \sum_{n=1}^{\infty} E_n X.$$

Moreover, for each $X \in D(A)$,

$$AX = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle X, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j X.$$

Therefore, for all $Z := \begin{pmatrix} X \\ Y \end{pmatrix} \in D(L) = D(A) \times L^2(\Omega, \mathscr{H})$, we obtain the following:

$$\begin{aligned}
L(t)Z &= \begin{pmatrix} 0 & I_{L^2(\Omega,\mathbb{H})} \\ -b(t)A & -a(t)I_{L^2(\Omega,\mathbb{H})} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\
&= \begin{pmatrix} Y \\ -b(t)AX - a(t)Y \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} E_n Y \\ -b(t) \sum_{n=1}^{\infty} \lambda_n E_n X - a(t) \sum_{n=1}^{\infty} E_n Y \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ -b(t)\lambda_n & -a(t) \end{pmatrix} \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\
&= \sum_{n=1}^{\infty} A_n(t) P_n Z,
\end{aligned}$$

where

$$P_n := \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix}, \; n \geq 1,$$

and

$$A_n(t) := \begin{pmatrix} 0 & 1 \\ -b(t)\lambda_n & -a(t) \end{pmatrix}, \; n \geq 1. \tag{7.15}$$

Now, the characteristic equation for $A_n(t)$ is given by

$$\lambda^2 + a(t)\lambda + \lambda_n b(t) = 0 \tag{7.16}$$

with discriminant given by $\Delta_n(t) = a^2(t) - 4\lambda_n b(t)$ for all $t \in \mathbb{R}$.

We suppose that there exists $\delta_0 > 0$ such that

$$\inf_{t\in\mathbb{R}} a(t) > 2\delta_0 > 0. \tag{7.17}$$

From Eq. (7.17) it easily follows that all the roots of Eq. (7.16) are nonzero (with nonzero real parts) given by

$$\lambda_1^n(t) = \frac{-a(t) + \sqrt{\Delta_n(t)}}{2} \quad \text{and} \quad \lambda_2^n(t) = \frac{-a(t) - \sqrt{-\Delta_n(t)}}{2},$$

that is,

$$\sigma(A_n(t)) = \left\{\lambda_1^n(t), \lambda_2^n(t)\right\}.$$

In view of the above, it is easy to see that there exist $\gamma_0 \geq 0$ and $\theta \in \left(\frac{\pi}{2}, \pi\right)$ such that

$$S_\theta \cup \left\{0\right\} \subset \rho\left(L(t) - \gamma_0 I\right)$$

for each $t \in \mathbb{R}$ where

$$S_\theta = \left\{z \in \mathbb{C} \setminus \{0\} : \left|\arg z\right| \leq \theta\right\}.$$

On the other hand, one can show without difficulty that $A_n(t) = K_n^{-1}(t) J_n(t) K_n(t)$, where $J_n(t), K_n(t)$, and $K_n^{-1}(t)$ are respectively given by

$$J_n(t) = \begin{pmatrix} \lambda_1^n(t) & 0 \\ 0 & \lambda_2^n(t) \end{pmatrix}, \quad K_n(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1^n(t) & \lambda_2^n(t) \end{pmatrix},$$

and

$$K_n^{-1}(t) = \frac{1}{\lambda_1^n(t) - \lambda_2^n(t)} \begin{pmatrix} -\lambda_2^n(t) & 1 \\ \lambda_1^n(t) & -1 \end{pmatrix}.$$

For $\lambda \in S_\theta$ and $Z \in L^2(\Omega, \mathscr{H})$, one has

$$\begin{aligned}
R(\lambda, L)Z &= \sum_{n=1}^{\infty} (\lambda - A_n(t))^{-1} P_n Z \\
&= \sum_{n=1}^{\infty} K_n(t) P_n (\lambda - J_n(t) P_n)^{-1} K_n^{-1}(t) P_n Z.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{E}\Big\|R(\lambda, L)Z\Big\|^2 &\leq \sum_{n=1}^{\infty} \Big\|K_n(t)P_n(\lambda - J_n(t)P_n)^{-1} K_n^{-1}(t)P_n\Big\|^2_{B(\mathscr{H})} \mathbf{E}\Big\|P_n Z\Big\|^2 \\
&\leq \sum_{n=1}^{\infty} \Big\|K_n(t)P_n\Big\|^2_{B(\mathscr{H})} \Big\|(\lambda - J_n(t)P_n)^{-1}\Big\|^2_{B(\mathscr{H})} \Big\|K_n^{-1}(t)P_n\Big\|^2_{B(\mathscr{H})} \mathbf{E}\Big\|P_n Z\Big\|^2.
\end{aligned}$$

Moreover, for $Z := \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in L^2(\Omega, \mathscr{H})$, we obtain

$$\begin{aligned}
\mathbf{E}\big\|K_n(t)P_n Z\big\|^2 &= \mathbf{E}\big\|E_n Z_1 + E_n Z_2\big\|^2 + \mathbf{E}\big\|\lambda_1^n E_n Z_1 + \lambda_2^n E_n Z_2\big\|^2 \\
&\leq 3\Big(1 + |\lambda_n^1(t)|^2\Big)\mathbf{E}\big\|Z\big\|^2.
\end{aligned}$$

Thus, there exists $C_1 > 0$ such that

$$\mathbf{E}\big\|K_n(t)P_n Z\big\|^2 \leq C_1 \big|\lambda_n^1(t)\big| \mathbf{E}\big\|Z\big\|^2 \quad \text{for all} \;\; n \geq 1.$$

Similarly, for $Z := \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in L^2(\Omega, \mathscr{H})$, one can show that there is $C_2 > 0$ such that

$$\mathbf{E}\Big\|K_n^{-1}(t)P_n Z\Big\|^2 \leq \frac{C_2}{\Big|\lambda_n^1(t)\Big|} \mathbf{E}\Big\|Z\Big\|^2 \quad \text{for all} \;\; n \geq 1.$$

Now, for $Z \in L^2(\Omega, \mathscr{H})$, we have

$$\begin{aligned}
\mathbf{E}\Big\|(\lambda - J_n(t)P_n)^{-1} Z\Big\|^2 &= \mathbf{E}\left\|\begin{pmatrix} \frac{1}{\lambda - \lambda_n^1(t)} & 0 \\ 0 & \frac{1}{\lambda - \lambda_n^2} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right\|^2 \\
&\leq \frac{1}{\Big|\lambda - \lambda_n^1(t)\Big|^2} \mathbf{E}\Big\|Z_1\Big\|^2 + \frac{1}{\Big|\lambda - \lambda_n^2(t)\Big|^2} \mathbf{E}\Big\|Z_2\Big\|^2.
\end{aligned}$$

Let $\lambda_0 > 0$. Define the function

$$\eta_t(\lambda) := \frac{1+\left|\lambda\right|}{\left|\lambda - \lambda_n^2(t)\right|}.$$

It is clear that η_t is continuous and bounded on the closed set

$$\Sigma := \left\{\lambda \in \mathbb{C} : \ \left|\lambda\right| \le \lambda_0, \ \left|\arg\lambda\right| \le \theta\right\}.$$

On the other hand, it is clear that η is bounded for $\left|\lambda\right| > \lambda_0$. Thus, the function η is bounded on S_θ. If we take

$$N = \sup\left\{\frac{1+\left|\lambda\right|}{\left|\lambda - \lambda_n^j(t)\right|} \ : \ \lambda \in S_\theta, n \ge 1; \ j = 1,2, \right\},$$

then

$$\mathbf{E}\left\|(\lambda - J_n(t)P_n)^{-1}Z\right\|^2 \le \frac{N}{1+\left|\lambda\right|}\mathbf{E}\left\|Z\right\|^2, \quad \lambda \in S_\theta.$$

Consequently,

$$\left\|R(\lambda, L(t))\right\| \le \frac{K}{1+\left|\lambda\right|}$$

for all $\lambda \in S_\theta$.

First of all, note that the domain $D = D(L(t))$ is independent of t. Thus, to check that Eq. (2.39) is satisfied it is enough to check that Eq. (2.41) holds. For that, note that the operator $L(t)$ is invertible with

$$L(t)^{-1} = \begin{pmatrix} -a(t)b^{-1}(t)A^{-1} & -b^{-1}(t)A^{-1} \\ \\ I_{\mathbb{H}} & 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Hence, for $t,s,r \in \mathbb{R}$, computing $\Big(L(t) - L(s)\Big)L(r)^{-1}$ and assuming that there exist $L_a, L_b \ge 0$ and $\mu \in (0,1]$ such that

$$\left|a(t) - a(s)\right| \le L_a\left|t-s\right|^\mu \ \text{ and } \ \left|b(t) - b(s)\right| \le L_b\left|t-s\right|^\mu, \tag{7.18}$$

it easily follows that there exists $C > 0$ such that

$$\mathbf{E}\left\|(L(t) - L(s))L(r)^{-1}Z\right\|^2 \le C\left|t-s\right|^{2\mu}\mathbf{E}\left\|Z\right\|^2.$$

In summary, the family of operators $\left\{L(t)\right\}_{t\in\mathbb{R}}$ satisfies Acquistapace–Terreni conditions. Consequently, there exists an evolution family $U(t,s)$ associated with it. Let us now check that $U(t,s)$ has exponential dichotomy. First of all, note that for every $t \in \mathbb{R}$, the family of linear operators $L(t)$ generates an analytic semigroup $(e^{\tau L(t)})_{\tau \ge 0}$ on $L^2(\Omega, \mathscr{H})$ given by

$$e^{\tau L(t)}Z = \sum_{l=1}^{\infty} K_l(t)^{-1}P_l e^{\tau J_l} P_l K_l(t) P_l Z,\ Z \in L^2(\Omega, \mathscr{H}).$$

On the other hand, we have

$$\mathbf{E}\Big\|e^{\tau L(t)}Z\Big\|^2 = \sum_{l=1}^{\infty} \Big\|K_l(t)^{-1}P_l\Big\|^2_{B(\mathscr{H})} \Big\|e^{\tau J_l}P_l\Big\|^2_{B(\mathscr{H})} \Big\|K_l(t)P_l\Big\|^2_{B(\mathscr{H})} \mathbf{E}\Big\|P_l Z\Big\|^2,$$

with for each $Z = \binom{Z_1}{Z_2}$,

$$\begin{aligned}
\mathbf{E}\Big\|e^{\tau J_l}P_l Z\Big\|^2 &= \left\|\begin{pmatrix} e^{\rho_1^l \tau}E_l & 0 \\ 0 & e^{\rho_2^l \tau}E_l \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right\|^2 \\
&\leq \mathbf{E}\Big\|e^{\rho_1^l \tau}E_l Z_1\Big\|^2 + \mathbf{E}\Big\|e^{\rho_2^l \tau}E_l Z_2\Big\|^2 \\
&\leq e^{-2\delta_0 \tau}\mathbf{E}\Big\|Z\Big\|^2.
\end{aligned}$$

Therefore,

$$\|e^{\tau L(t)}\| \leq Ce^{-\delta_0 \tau}, \quad \tau \geq 0. \tag{7.19}$$

Using the continuity of a, b and the equality

$$R(\lambda, L(t)) - R(\lambda, L(s)) = R(\lambda, L(t))(L(t) - L(s))R(\lambda, L(s)),$$

it follows that the mapping $J \ni t \mapsto R(\lambda, L(t))$ is strongly continuous for $\lambda \in S_\omega$ where $J \subset \mathbb{R}$ is an arbitrary compact interval. Therefore, $L(t)$ satisfies the assumptions of [160, Corollary 2.3], and thus the evolution family $(U(t,s))_{t \geq s}$ is exponentially stable.

It remains to verify that $R(\gamma_0, L(\cdot)) \in AP(\mathbb{R}, B(L^2(\Omega; \mathscr{H})))$. For that we need to show that $L^{-1}(\cdot) \in AP(\mathbb{R}, B(L^2(\Omega, \mathscr{H})))$. Since $t \to a(t),\ t \to b(t)$, and $t \to b(t)^{-1}$ are almost periodic it follows that $t \to d(t) = -\dfrac{a(t)}{b(t)}$ is almost periodic, too. So for all $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a τ such that

$$\Big|\frac{1}{b(t+\tau)} - \frac{1}{b(t)}\Big| < \frac{\varepsilon}{\|A^{-1}\|\sqrt{2}}, \quad \Big|d(t+\tau) - d(t)\Big| < \frac{\varepsilon}{\|A^{-1}\|\sqrt{2}}$$

for all $t \in \mathbb{R}$.

Clearly,

$$\begin{aligned}
\Big\|L^{-1}(t+\tau) - L^{-1}(t)\Big\| &\leq \left(\Big|\frac{1}{b(t+\tau)} - \frac{1}{b(t)}\Big|^2 + \Big|d(t+\tau) - d(t)\Big|^2\right)^{1/2} \|A^{-1}\|_{B(\mathbb{H})} \\
&< \varepsilon
\end{aligned}$$

and hence $t \to L^{-1}(t)$ is almost periodic with respect to $L^2(\Omega,\mathscr{H})$-operator-topology. Therefore, $R(\gamma_0, L(\cdot)) \in AP(\mathbb{R}, B(L^2(\Omega;\mathscr{H})))$.

To study the existence of square-mean almost periodic solutions of Eq. (7.14), we use the general results obtained in Section 6.3.

Remark 7.1. Note that it follows from $(7H)_5$ that $F_i(i=1,2) : \mathbb{R} \times L^2(\Omega;\mathbb{H}) \to L^2(\Omega;\mathbb{H})$ is square-mean almost periodic. Furthermore, $X \mapsto F_i(t,X)$ is uniformly continuous on any bounded subset K of $L^2(\Omega;\mathbb{H})$ for each $t \in \mathbb{R}$. Finally,

$$\sup_{t\in\mathbb{R}} \mathbf{E}\big\|F_i(t,X)\big\|^2 \le \mathscr{M}_i\big(\|X\|_\infty\big)$$

where $\mathscr{M}_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, monotone increasing function satisfying

$$\lim_{r\to\infty} \frac{\mathscr{M}_i(r)}{r} = 0\,.$$

Theorem 7.2. *Suppose assumptions* $(7H)_4$ and $(7H)_5$ *hold, then the nonautonomous differential equation* (7.14) *has at least one square-mean almost periodic mild solution.*

Proof. In view of Remark 7.1, the proof follows along the same lines as that of Theorem 6.4 and hence omitted.

7.3 Bibliographical Notes

The results of this chapter are mainly inspired by the recent work of Diagana [55]. Although an *n*-order version of this chapter is known in the deterministic case (see Diagana [57, 56]), a stochastic version of it is, to the best of the authors' knowledge, unknown.

Chapter 8
Mean Almost Periodic Solutions to Some Stochastic Difference Equations

8.1 Introduction

This chapter deals with discrete-time stochastic processes known as random sequences. Here, we are particularly interested in the study of almost periodicity of those random sequences and their applications to stochastic difference equations. In biology for instance, fluctuations in nature are hardly periodic although one can deliberately periodically fluctuate environment parameters in controlled laboratory experiments. That is, almost periodicity is more likely to accurately describe natural fluctuations.

In this chapter, $(\mathscr{B}, \|\cdot\|)$ denotes a Banach space and $\mathbb{Z}_+$ the set of all nonnegative integers.

8.2 Basic Definitions

In this section we develop a basic theory for mean almost periodic random sequences on $\mathbb{Z}_+$. To facilitate our task, we first introduce the notations needed in the sequel.

Define $L^1(\Omega;\mathscr{B})$ to be the space of all $\mathscr{B}$-valued random variables V such that $\mathbf{E}\|V\| < \infty$. It is then routine to check that $L^1(\Omega;\mathscr{B})$ is a Banach space when it is equipped with its natural norm $\|\cdot\|_1$ defined by $\|V\|_1 := \mathbf{E}\|V\|$ for each $V \in L^1(\Omega,\mathscr{B})$.

Let $X = \{X_n\}_{n\in\mathbb{Z}_+}$ be a sequence of $\mathscr{B}$-valued random variables satisfying $\mathbf{E}\|X_n\| < \infty$ for each $n \in \mathbb{Z}_+$. Thus, interchangeably we can, and do, speak of such a sequence as a function, which goes from $\mathbb{Z}_+$ into $L^1(\Omega;\mathscr{B})$.

This setting requires the following preliminary definitions.

Definition 8.1. [132] An $L^1(\Omega;\mathscr{B})$-valued random sequence $X = \{X(n)\}_{n\in\mathbb{Z}_+}$ is said to be *stochastically bounded* whenever

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$$\lim_{N\to\infty}\left(\sup_{n\in\mathbb{Z}_+}\mathbf{P}\Big\{\omega : \|X(\omega,n)\| > N\Big\}\right) = 0.$$

Definition 8.2. An $L^1(\Omega;\mathscr{B})$-valued random sequence $X = \{X(n)\}_{n\in\mathbb{Z}_+}$ is said to be *mean (Bohr) almost periodic* if for each $\varepsilon > 0$ there exists $N_0(\varepsilon) > 0$ such that among any N_0 consecutive integers there exists at least an integer $p > 0$ for which

$$\mathbf{E}\|X(n+p) - X(n)\| < \varepsilon, \forall n \in \mathbb{Z}_+.$$

An integer $p > 0$ with the above-mentioned property is called an ε-almost period for $X = \{X(n)\}_{n\in\mathbb{Z}_+}$. The collection of all those $\mathscr{B}$-valued random sequences $X = \{X(n)\}_{n\in\mathbb{Z}_+}$ which are mean (Bohr) almost periodic is then denoted by $AP(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$.

Similarly, one defines the mean (Bochner) almost periodicity as follows:

Definition 8.3. An $L^1(\Omega;\mathscr{B})$-valued random sequence $X = \{X(n)\}_{n\in\mathbb{Z}_+}$ is called *mean (Bochner) almost periodic* if for every sequence $\{h(n)\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}_+$ there exists a subsequence $\{h(k_s)\}_{s\in\mathbb{Z}_+}$ such that $\{X(n+h(k_s))\}_{s\in\mathbb{Z}_+}$ converges (in the mean) uniformly with respect to $n \in \mathbb{Z}_+$.

8.3 Preliminary Results

Theorem 8.1. *An $L^1(\Omega;\mathscr{B})$-valued random sequence $X = \{X(n)\}_{n\in\mathbb{Z}_+}$ is mean (Bochner) almost periodic if and only if it is mean (Bohr) almost periodic.*

Proof. The proof, with slight changes, follows along the same lines as the proof of [63, Theorem 2.4, p. 241]. However, for the sake of clarity, we reproduce it here.

First of all, let us show that if $x = \{x(t)\}_{t\in\mathbb{Z}_+}$ is Bochner almost periodic, then it is Bohr almost periodic. To achieve this, we show that if $x = \{x(t)\}_{t\in\mathbb{Z}_+}$ is not Bohr almost periodic, then it is not Bochner almost periodic.

Suppose that $x = \{x(t)\}_{t\in\mathbb{Z}_+}$ is not Bohr almost periodic. Then there exists at least one $\varepsilon > 0$ such that for any positive integer T_0, there exist T_0 consecutive positive integers which contain no ε-period related to the sequence $\{x(t)\}_{t\in\mathbb{Z}_+}$. Now, let $h(1) \in \mathbb{Z}_+$ and let $2\alpha_1+1, 2\alpha_1+2, 2\alpha_1+3, ..., 2\beta_1-2, 2\beta_1-1$ be $(2\beta_1-2\alpha_1-1)$-positive integers $(\alpha_1,\beta_1 \in \mathbb{Z}_+)$ such that $2\beta_1-2\alpha_1-2 > 2h(1)$ or $\beta_1-\alpha_1-1 > h(1)$ and the sequence $2\alpha_1+1, 2\alpha_1+2, 2\alpha_1+3, ..., 2\beta_1-2, 2\beta_1-1$ does not contain any ε-period related to $\{x(t)\}_{t\in\mathbb{Z}_+}$.

Next, let $h(2) = \frac{1}{2}(2\alpha_1+2\beta_1) = \alpha_1+\beta_1$. Clearly, $h(2)-h(1)$ is a (positive) integer such that $2\alpha_1+1 < h(2)-h(1) < 2\beta_1-1$, and hence $h(2)-h(1)$ cannot be an ε-period. Thus, there exist $2\alpha_2+1, 2\alpha_2+2, 2\alpha_2+3, ..., 2\beta_2-2, 2\beta_2-1$ such that $2\beta_2-2\alpha_2-2 > 2(h(1)+h(2))$, which does not contain any ε-period related to $\{x(t)\}_{t\in\mathbb{Z}_+}$. Setting $h(3) = \frac{1}{2}(2\alpha_2+2\beta_2) = \alpha_2+\beta_2$, it follows that $h(3)-h(2)$, $h(3)-h(1)$ are respectively one of the terms $2\alpha_2+1, 2\alpha_2+2, 2\alpha_2+3, ..., 2\beta_2-2, 2\beta_2-1$, and hence $h(3)-h(2)$, $h(3)-h(1)$ are not ε-period related to $\{x(t)\}_{t\in\mathbb{Z}_+}$.

Proceeding as previously, one defines the numbers $h(4), h(5), \ldots$, such that none of the expressions $h(i) - h(j)$ for $i > j$ is an ε-period for the sequence $\{x(t)\}_{t\in\mathbb{Z}_+}$.

Consequently, for all $i, j \in \mathbb{Z}_+$,

$$\begin{aligned}\sup_{i,j}\mathbf{E}\|x(t+h(i)) - x(t+h(j))\| &\geq \sup_{i>j}\mathbf{E}\|x(t+h(i)) - x(t+h(j))\| \\ &= \sup_{i>j}\mathbf{E}\|x(t+h(i)-h(j)) - x(t)\| \\ &\geq \varepsilon.\end{aligned}$$

Therefore, the sequence $\{x(t+h(i))\}_{i\in\mathbb{Z}_+}$ cannot contain any uniformly convergent sequence, and hence $\{x(t)\}_{t\in\mathbb{Z}_+}$ is not Bochner almost periodic.

Conversely, suppose that the sequence $\{x(t)\}_{t\in\mathbb{Z}_+}$ is Bohr almost periodic and $\{t_j\}_{j\in\mathbb{Z}_+}$ is a sequence of positive integers. Here, we adapt our proof to the one given in [88, Proof of Theorem 4.9]. For each $\varepsilon > 0$ there exists an integer $T_0 > 0$ such that between t_j and $T_0 + t_j$ there exists an ε-period τ_j with $0 \leq \tau_j - t_j \leq T_0$. Setting $s_j = \tau_j - t_j$, one can see that s_j can take only a finite number (at most $T_0 + 1$) values, and hence there is some s, $0 \leq s \leq T_0$ such that $s_j = s$ for an infinite numbers of j''s. Let these indexes be numbered as j_k, then we have

$$\mathbf{E}\|x(t+t_j) - x(t+s_j)\| = \mathbf{E}\|x(t+\tau_j+s_j) - x(t+s_j)\| < \varepsilon,$$

and hence,

$$\mathbf{E}\|x(t+t_j) - x(t+s_j)\| < \varepsilon$$

for all $t \in \mathbb{Z}_+$.

One may complete the proof by proceeding exactly as in [88, Proof of Theorem 4.9] and using [88, Proposition 4.7] relative to $\mathbb{Z}_+$ rather than $\mathbb{Z}$.

Now let $\{\varepsilon_r\}_{r\in\mathbb{Z}_+}$ be a sequence such that $\varepsilon \to 0$ as $r \to \infty$, say $\varepsilon_r = \dfrac{1}{r+1}$. Now, from the sequence $\{x(n+t_j)\}_{j\in\mathbb{Z}_+}$, consider a subsequence chosen so that

$$\mathbf{E}\|x(n+t_{j_i^1}) - x(n+s^1)\| \leq \varepsilon_1.$$

Next, from the previous sequence, we take a new subsequence such that

$$\mathbf{E}\|x(n+t_{j_i^2}) - x(n+s^2)\| \leq \varepsilon_2.$$

Repeating this procedure and for each $r \in Z_+$ we obtain a subsequence $\left\{x(n+t_{j_i^r})\right\}_{i\in\mathbb{Z}_+}$ such that

$$\mathbf{E}\|x(n+t_{j_i^r}) - x(n+s^r)\| \leq \varepsilon_r.$$

Now, for the diagonal sequence, $\left\{x(n+t_{j_i^i})\right\}_{i\in\mathbb{Z}_+}$, for each $\varepsilon > 0$ take $k(\varepsilon) \in \mathbb{Z}_+$ such that $\varepsilon_{k(\varepsilon)} < \frac{\varepsilon}{2}$, where ε_r belongs to the previous sequence $\{\varepsilon_r\}_{r\in\mathbb{Z}_+}$.

Using the fact that the sequences $\{t_{j_r^r}\}$ and $\{t_{j_s^s}\}$ are both subsequences of $\left\{t_{j_i^{k(\varepsilon)}}\right\}$, for $r \geq k(\varepsilon)$ we have

$$\begin{aligned}\mathbf{E}\|x(n+t_{j_r^r})-x(n+t_{j_s^s})\| &\leq \mathbf{E}\|x(n+t_{j_r^r})-x(n+s^k)\| \\ &+ \mathbf{E}\|x(n+s^k)-x(n+t_{j_s^s})\| \\ &\leq \varepsilon_{k(\varepsilon)}+\varepsilon_{k(\varepsilon)} \\ &\leq \varepsilon,\end{aligned}$$

and hence the sequence $\left\{x(n+t_{j_i^i})\right\}_{i\in\mathbb{Z}_+}$ is a Cauchy sequence.

An important and straightforward consequence of Theorem 8.1 is the next corollary, which plays a key role in the proof of Lemma 8.3.

Corollary 8.1. *If $X_1 = \{X^1(n)\}_{n\in\mathbb{Z}_+}$, $X_2 = \{X^2(n)\}_{n\in\mathbb{Z}_+}$, ..., and $X_N = \{X^N(n)\}_{n\in\mathbb{Z}_+}$ are N random sequences, which belong to $AP(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$, then for each $\varepsilon > 0$ there exists $N_0(\varepsilon) > 0$ such that among any $N_0(\varepsilon)$ consecutive integers there exists an integer $p > 0$ for which*

$$\mathbf{E}\|X^j(n+p)-X^j(n)\| < \varepsilon$$

for each $n \in \mathbb{Z}_+$ and for $j = 1,2,...,N$.

Definition 8.4. A sequence of $\mathscr{B}$-valued random variables $X = \{X(n)\}_{n\in\mathbb{Z}_+}$ is said to be *almost periodic in probability* if for each $\varepsilon > 0$, $\eta > 0$, there exists $N_0(\varepsilon) > 0$ such that among any N_0 consecutive integers there exists at least an integer $p > 0$ for which

$$\mathbf{P}\{\omega : \|X(\omega,n+p)-X(\omega,n)\| \geq \varepsilon\} < \eta, \forall n \in \mathbb{Z}_+.$$

This definition of almost periodicity in probability is similar to the concept of (Bohr) almost periodicity on $\mathbb{R}_+$.

Lemma 8.1. *If X belongs to $AP(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$, then*
(i) there exists a constant $M > 0$ such that $\mathbf{E}\|X(n)\| \leq M$ for each $n \in \mathbb{Z}_+$;
(ii) X is stochastically bounded; and
(iii) X is almost periodic in probability.

Proof. (i) One follows along the same lines as in the proof of [63, Lemma 2.6]. Assume that $\{\mathbf{E}\|X(n)\|\}_{n\in\mathbb{Z}_+}$ is not bounded. Then for some subsequence $\mathbf{E}\|X(n_i)\| \to \infty$ *as* $i \to \infty$. Let $\varepsilon = 1$. Then there exists an integer $N_0(\varepsilon) > 0$ that satisfies the almost periodicity definition. There exists $n_i = s_1$ such that $n_i = s_1 > N_0(\varepsilon)$. Then among the integers

$$\{s_1 - N_0(\varepsilon)+1, s_1 - N_0(\varepsilon)+2, ..., s_1\}$$

there exists $\widehat{s_1}$ such that

$$\mathbf{E}\|X(n+\widehat{s}_1)-X(n)\|<1.$$

Next, choose $n_j = s_2$ such that $n_j = s_2 > N_0(\varepsilon)+s_1$. Then among the integers

$$\{s_2 - N_0(\varepsilon)+1, s_2 - N_0(\varepsilon)+2, .., s_2\}$$

there exists $\widehat{s}_2$ such that

$$\mathbf{E}\|X(n+\widehat{s}_2)-X(n)\|<1.$$

Repeating this process, we obtain a sequence $\{\widehat{s}_i\}\to\infty$ as $i\to\infty$ such that

$$\mathbf{E}\|X(n+\widehat{s}_i)-X(n)\|<1 \text{ for } r=1,2,3,...,$$

and a subsequence $\{s_i\}$ of $\{n_i\}$ with $\{s_i\}\to\infty$ as $i\to\infty$. Moreover,

$$s_i = \widehat{s}_i + u_i$$

where $0\le u_i < N_0(\varepsilon)$.

Since $\{u_i\}$ is finite, there exists u_{i_0} that is repeated infinitely many times and $s_{i_r} = \widehat{s}_{i_r} + u_{i_0}$, where $i_r\to\infty$ as $i\to\infty$. Therefore,

$$\mathbf{E}\|X(n+\widehat{s}_{i_r})-X(u_{i_0})\|<1.$$

Moreover,

$$\mathbf{E}\|X(n+s_{i_r})-X(u_{i_0})\|<1.$$

Hence, $\{X(s_{i_r})\}$ is bounded; a contradiction.

To prove (ii), we use the Markov Inequality to obtain

$$\sup_{n\in\mathbb{Z}_+}\mathbf{P}\Big\{\omega:\|X(\omega,n)\|>N\Big\}\le\frac{1}{N}\sup_{n\in\mathbb{Z}_+}\mathbf{E}\|X(n)\|\le\frac{M}{N},$$

and hence

$$\lim_{N\to\infty}\left(\sup_{n\in\mathbb{Z}_+}\mathbf{P}\Big\{\omega:\|X(\omega,n)\|>N\Big\}\right)=0.$$

Using similar arguments, we also obtain the almost periodicity in probability of X.

Let $UB(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$ denote the collection of all uniformly bounded $L^1(\Omega;\mathscr{B})$-valued random sequences $X=\{X(n)\}_{n\in\mathbb{Z}_+}$. It is then easy to check that the space $UB(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$ is a Banach space when it is equipped with the norm

$$\|X\|_\infty = \sup_{n\in\mathbb{Z}_+}\mathbf{E}\|X(n)\|.$$

Lemma 8.2. *$AP(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))\subset UB(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$ is a closed space.*

Proof. It is clear that $AP(\mathbb{Z}_+;L^1(\Omega;\mathscr{B})) \subset UB(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$ (see (i) of Lemma 8.1). Now let $(X_m)_{m\in\mathbb{N}} \subset AP(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$ be a random sequence such that $\|X_m - X\|_\infty \mapsto 0$ as $m \mapsto \infty$ for some $X \in UB(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$. To complete the proof we have to prove that $X \in AP(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$.

Since X is uniformly bounded in the sense of $L^1(\Omega;\mathscr{B})$, it remains to prove that it is mean almost periodic. Now, let $\varepsilon > 0$ and choose m such that

$$\|X_m - X\|_\infty < \frac{\varepsilon}{3}.$$

Now since $(X_m)_{m\in\mathbb{N}}$ is mean almost periodic, then there exists a positive integer $N_0(\varepsilon)$ such that among any N_0 consecutive integers, there exists at least an integer $p > 0$ for which

$$\mathbf{E}\|X_m(n+p) - X_m(n)\| < \frac{\varepsilon}{3}, \ \forall n \in \mathbb{Z}_+.$$

Now

$$\begin{aligned}
\mathbf{E}\|X(n+p) - X(n)\| &\le \mathbf{E}\|X_m(n+p) - X(n+p)\| \\
&+ \mathbf{E}\|X_m(n+p) - X_m(n)\| \\
&+ \mathbf{E}\|X_m(n) - X(n)\| \\
&\le \mathbf{E}\|X_m(n+p) - X_m(n)\| \\
&+ 2\sup_{n\in\mathbb{Z}_+} \mathbf{E}\|X_m(n) - X(n)\| \\
&< 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon,
\end{aligned}$$

and hence

$$\sup_{n\in\mathbb{Z}_+} \mathbf{E}\|X(n+p) - X(n)\| \le \varepsilon.$$

In view of the above, the space $AP(\mathbb{Z}_+;L^1(\Omega;\mathscr{B}))$ of random mean almost periodic sequences equipped with the sup norm $\|\cdot\|_\infty$ is also a Banach space.

8.4 Mean Almost Periodic Solutions to Stochastic Beverton–Holt Equations

In constant environments, theoretical discrete-time population models are usually formulated under the assumption that the dynamics of the total population size in generation n, denoted by $x(n)$, are governed by equations of the form

$$x(n+1) = f(x(n)) + \gamma x(n), \tag{8.1}$$

where $\gamma \in (0,1)$ is the constant "probability" of surviving per generation, and the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ models the birth or recruitment process.

Almost periodic effects can be introduced into (8.1) by writing the recruitment function or the survival probability as almost periodic sequences. This is modeled with the equation

$$x(n+1) = f(n,x(n)) + \gamma_n x(n), \tag{8.2}$$

where either $\{\gamma_n\}_{n\in\mathbb{Z}_+}$ or $\{f(n,x(n))\}_{n\in\mathbb{Z}_+}$ are almost periodic and $\gamma_n \in (0,1)$.

In a recent paper, Franke and Yakubu [74] studied (8.2) with the periodic Beverton–Holt recruitment function

$$f(n,x(n)) = \frac{(1-\gamma_n)\mu K_n x(n)}{(1-\gamma_n)K_n + (\mu - 1 + \gamma_n)x(n)}, \tag{8.3}$$

where the carrying capacity K_n is p-periodic, $K_{n+p} = K_n$ for all $t \in \mathbb{Z}_+$ and $\mu > 1$.

We now introduce the notations needed in the sequel. From now on we assume that both the carrying capacity K_n and the survival rate γ_n are random and that γ_n, $n \in \mathbb{Z}_+$ are independent and independent of the sequence $\{K_n\}_{n\in\mathbb{Z}_+}$. Let $\mathscr{B} = \mathbb{R}_+ = [0,+\infty)$ equipped with the absolute value of $\mathbb{R}$.

In the present work we investigate the stochastic nonautonomous Beverton–Holt equations (8.2)–(8.3). It is shown (Theorem 8.2) that under some suitable assumptions, where both $\{K_n\}_{n\in Z_+}$ and $\{\gamma_n\}_{n\in Z_+}$ belong to $AP(\mathbb{Z}_+;L^1(\Omega;\mathbb{R}_+))$ and $\mu > 1$, (8.2)–(8.3) have a unique mean almost periodic solution on $\mathbb{Z}_+$.

We now state our main theorem.

Theorem 8.2. *Suppose that both sequences $\{K_n\}_{n\in Z_+}$ and $\{\gamma_n\}_{n\in Z_+}$ belong to $AP(\mathbb{Z}_+;L^1(\Omega;\mathbb{R}_+))$ and $\mu > 1$. Then Eqs.* (8.2)–(8.3) *have a unique random mean almost periodic solution whenever*

$$\sup_{n\in\mathbb{Z}_+} \{\mathbf{E}[\gamma_n]\} < \frac{1}{\mu+1}.$$

The proof of Theorem 8.2 requires the following lemma.

Lemma 8.3. *Let*

$$f(n,X(n)) = \frac{(1-\gamma_n)\mu K_n X}{(1-\gamma_n)K_n + (\mu - 1 + \gamma_n)X(n)}$$

where both $\{K_n\}_{n\in Z^+}$ and $\{\gamma_n\}_{n\in Z_+}$ belong to $AP(\mathbb{Z}_+;L^1(\Omega;\mathbb{R}_+))$ and $\mu > 1$. Then,

(i) f is μ-Lipschitz in the following sense:

$$\mathbf{E}|f(n,U) - f(n,V)| \le \mu\ \mathbf{E}|U - V|,\ \ \forall U,V \in L^1(\Omega;\mathbb{R}_+),\ n \in \mathbb{Z}_+;$$

(ii) If X belongs to $AP(\mathbb{Z}_+;L^1(\Omega;\mathbb{R}_+))$, then the sequence $\{f(n,X(n))\}_{n\in\mathbb{Z}_+}$ also belongs to $AP(\mathbb{Z}_+;L^1(\Omega;\mathbb{R}_+))$.

Proof. (i) It is routine to check that $|f(n,U) - f(n,V)| \le \mu|U - V|$, and hence $\mathbf{E}|f(n,U) - f(n,V)| \le \mu\ \mathbf{E}|U - V|$.

To prove (ii), set $A_n = (1-\gamma_n)K_n$ and $B_n = \mu - 1 + \gamma_n$. Then f can be written as follows:

$$f(n,X(n)) = \mu \, \frac{A_n X(n)}{A_n + B_n X(n)} \quad \text{for each } \; n \in \mathbb{Z}_+.$$

Using the fact that $\{\gamma_n\}$, $\{K_n\}$, and $\{X(n)\}$ are mean almost periodic and making use of respectively Lemma 8.1(i) and Corollary 8.1, we can choose a constant $M > 0$ such that $\mathbf{E}|K_n| < M$ for all $n \in \mathbb{Z}_+$ and for each $\varepsilon > 0$ there exists a positive integer $N_0(\varepsilon)$ such that among any $N_0(\varepsilon)$ consecutive integers, there exists an integer $p > 0$, a common ε-almost period for $\{\gamma_n\}$, $\{K_n\}$, and $\{X(n)\}$, for which

$$\mathbf{E}|\gamma_{n+p} - \gamma_n| \le \frac{\varepsilon(\mu-1)^2}{3\mu^2 M}\,, \;\; \mathbf{E}|K_{n+p} - K_n| \le \frac{\varepsilon(\mu-1)}{6\mu^2}\,,$$

and

$$\mathbf{E}|X(n+p) - X(n)| \le \frac{\varepsilon}{6\mu}$$

for all $n \in \mathbb{Z}_+$.

We now evaluate $|f(n+p,X(n+p)) - f(n,X(n))|$. We have

$$\begin{aligned}
&|f(n+p,X(n+p)) - f(n,X(n))| \\
&\le \mu \left| \frac{A_{n+p}X(n+p)}{A_{n+p} + B_{n+p}X(n+p)} - \frac{A_{n+p}X(n)}{A_{n+p} + B_{n+p}X(n+p)} \right| \\
&\quad + \mu \left| \frac{A_{n+p}X(n)}{A_{n+p} + B_{n+p}X(n+p)} - \frac{A_{n+p}X(n)}{A_{n+p} + B_{n+p}X(n)} \right| \\
&\quad + \mu \left| \frac{A_{n+p}X(n)}{A_{n+p} + B_{n+p}X(n)} - \frac{A_n X(n)}{A_n + B_n X(n)} \right| \\
&\le \mu \, |X(n+p) - X(n)| + \mu \, A_{n+p}X(n) \\
&\quad \times \left| \frac{1}{A_{n+p} + B_{n+p}X(n+p)} - \frac{1}{A_{n+p} + B_{n+p}X(n)} \right| \\
&\quad + \mu \left| \frac{A_{n+p}X(n)}{A_{n+p} + B_{n+p}X(n)} - \frac{A_n X(n)}{A_n + B_n X(n)} \right| \\
&\le \mu \, |X(n+p) - X(n)| \\
&\quad + \mu \frac{A_{n+p}}{A_{n+p} + B_{n+p}X(n+p)} \cdot \frac{B_{n+p}X(n)}{A_{n+p} + B_{n+p}X(n)} \, |X(n+p) - X(n)| \\
&\quad + \mu \left| \frac{A_{n+p}X(n)}{A_{n+p} + B_{n+p}X(n)} - \frac{A_n X(n)}{A_n + B_n X(n)} \right| .
\end{aligned}$$

But

$$\begin{aligned}\left|\frac{A_{n+p}X(n)}{A_{n+p}+B_{n+p}X(n)}-\frac{A_nX(n)}{A_n+B_nX(n)}\right| &\leq \mu\left|\frac{(A_{n+p}B_n-A_nB_{n+p})X(n)^2}{B_{n+p}B_nX(n)^2}\right| \\ &= \mu\left|\frac{A_{n+p}}{B_{n+p}}-\frac{A_n}{B_n}\right|.\end{aligned}$$

Thus,

$$|f(n+p,X(n+p))-f(n,X(n))| \leq 2\mu|X(n+p)-X(n)|+\mu\left|\frac{A_{n+p}}{B_{n+p}}-\frac{A_n}{B_n}\right|,$$

which in turn implies that

$$\mathbf{E}|f(n+p,X(n+p))-f(n,X(n))| \leq 2\mu\mathbf{E}|X(n+p)-X(n)|+\mu\mathbf{E}\left|\frac{A_{n+p}}{B_{n+p}}-\frac{A_n}{B_n}\right|.$$

We now evaluate carefully $\mathbf{E}\left|\frac{A_{n+p}}{B_{n+p}}-\frac{A_n}{B_n}\right|$ using the hypothesis of independence of the random sequence $\{\gamma_n\}_{n\in\mathbb{Z}_+}$. We have

$$\begin{aligned}
&\mathbf{E}\left|\frac{A_{n+p}}{B_{n+p}}-\frac{A_n}{B_n}\right| = \mathbf{E}\left|\frac{(1-\gamma_{n+p})K_{n+p}}{\mu-1+\gamma_{n+p}}-\frac{(1-\gamma_n)K_n}{\mu-1+\gamma_n}\right| \\
&= \mathbf{E}[\frac{1}{(\mu-1+\gamma_{n+p})(\mu-1+\gamma_n)}\;|(\mu-1)[K_{n+p}-K_n]-\gamma_n\gamma_{n+p}[K_{n+p}-K_n] \\
&\quad -(\mu-1)[\gamma_{n+p}K_{n+p}-\gamma_nK_n]+[\gamma_nK_{n+p}-\gamma_{n+p}K_n]|] \\
&= \mathbf{E}[\frac{1}{(\mu-1+\gamma_{n+p})(\mu-1+\gamma_n)}\;|(\mu-1)[K_{n+p}-K_n]-\gamma_n\gamma_{n+p}[K_{n+p}-K_n] \\
&\quad -(\mu-1)K_{n+p}\,[\gamma_{n+p}-\gamma_n]+\gamma_n\,[K_{n+p}-K_n]+\gamma_n\,[K_{n+p}-K_n]-[\gamma_{n+p}-\gamma_n]|] \\
&= \mathbf{E}[|\frac{\mu-1}{(\mu-1+\gamma_{n+p})(\mu-1+\gamma_n)}[K_{n+p}-K_n] \\
&\quad -\frac{\gamma_n\gamma_{n+p}}{(\mu-1+\gamma_{n+p})(\mu-1+\gamma_n)}[K_{n+p}-K_n] \\
&\quad -\frac{\mu-1}{(\mu-1+\gamma_{n+p})(\mu-1+\gamma_n)}K_{n+p}\,[\gamma_{n+p}-\gamma_n] \\
&\quad +\frac{(\mu-1)\gamma_n}{(\mu-1+\gamma_{n+p})(\mu-1+\gamma_n)}[K_{n+p}-K_n] \\
&\quad -\frac{\gamma_n}{(\mu-1+\gamma_{n+p})(\mu-1+\gamma_n)}[K_{n+p}-K_n] \\
&\quad -\frac{1}{(\mu-1+\gamma_{n+p})(\mu-1+\gamma_n)}K_n\,[\gamma_{n+p}-\gamma_n]| \\
&\leq \frac{1}{\mu-1}\mathbf{E}|K_{n+p}-K_n|+\mathbf{E}|K_{n+p}-K_n|+\frac{1}{\mu-1}\mathbf{E}|K_{n+p}|\,\mathbf{E}|\gamma_{n+p}-\gamma_n|
\end{aligned}$$

$$+\mathbf{E}|K_{n+p}-K_n|+\frac{1}{\mu-1}\mathbf{E}|K_{n+p}-K_n|+\frac{1}{(\mu-1)^2}\mathbf{E}|K_n|\,\mathbf{E}|\gamma_{n+p}-\gamma_n|$$

$$\leq \frac{2\mu}{\mu-1}\mathbf{E}|K_{n+p}-K_n|]+\frac{\mu}{(\mu-1)^2}M\cdot\mathbf{E}|\gamma_{n+p}-\gamma_n|\,.$$

By combining, we obtain

$$\begin{aligned}\mathbf{E}|f(n+p,X(n+p))-f(n,X(n))| &\leq 2\mu\,\mathbf{E}|X(n+p)-X(n)|\\ &+\frac{2\mu^2}{\mu-1}\mathbf{E}|K_{n+p}-K_n|\\ &+\left(\frac{\mu}{(\mu-1)}\right)^2 M\cdot\mathbf{E}|\gamma_{n+p}-\gamma_n|\\ &\leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\,.\end{aligned}$$

We now prove Theorem 8.2.

Proof. By Lemma 8.3(ii), if $u\in AP(\mathbb{Z}_+,L^1(\Omega;\mathbb{R}_+)$, then $n\to f(n,u(n))$ belongs to $AP(\mathbb{Z}_+,L^1(\Omega;\mathbb{R}_+))$. Define the nonlinear operator Γ by setting

$$\Gamma: AP(\mathbb{Z}_+,L^1(\Omega;\mathbb{R}_+))\mapsto AP(\mathbb{Z}_+,L^1(\Omega;\mathbb{R}_+)),$$

where

$$\Gamma u(n):=\sum_{r=0}^{n-1}\left(\prod_{s=r}^{n-1}\gamma_s\right)f(r,u(r)).$$

It is clear that Γ is well defined. Now, let $u,v\in AP(\mathbb{Z}_+,L^1(\Omega;\mathbb{R}_+))$ having the same property as x defined in the Beverton–Holt equation. Since $\{\gamma_n, n\in\mathbb{Z}_+\}$ are independent and independent of u and v, one can easily see that

$$\mathbf{E}\,|\Gamma u(n)-\Gamma v(n)|\leq\sum_{r=0}^{n-1}\left\{\left(\prod_{s=r}^{n-1}\mathbf{E}|\gamma_s|\right)\mathbf{E}\,|f(r,u(r))-f(r,v(r))|\right\},$$

and hence letting $\beta=\sup_{n\in\mathbb{Z}_+}\mathbf{E}[\gamma_n]$ we obtain

$$\sup_{n\in\mathbb{Z}_+}\mathbf{E}\,|\Gamma u(n)-\Gamma v(n)|\leq\left(\frac{\mu\beta}{1-\beta}\right)\sup_{n\in\mathbb{Z}_+}\mathbf{E}\,|u(n)-v(n)|\,.$$

Obviously, Γ is a contraction whenever $\dfrac{\mu\beta}{1-\beta}<1$. In that event, using the Banach fixed point theorem it easily follows that Γ has a unique fixed point, $\bar{x}$, which obviously is the unique mean almost periodic solution of Eqs. (8.2)–(8.3).

8.5 Bibliographical Notes

All the main results presented in this chapter are based on some recent work by Bezandry, Diagana and Elaydi [24] and Diagana, Elaydi and Yakubu [63].

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